Optimal Stopping with Random Maturity under Nonlinear Expectations

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Abstract

We analyze an optimal stopping problem $\sup_{\gamma\in\mathcal{T}} \overline{\mathscr{E}}_0\big[\mathscr{Y}_{\gamma\wedge\tau_0}\big]$ with random maturity τ_0 under a nonlinear expectation $\overline{\mathscr{E}}_0[\cdot] := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}[\cdot]$, where $\mathcal{P}$ is a weakly compact set of mutually singular probabilities. The maturity τ_0 is specified as the hitting time to level 0 of some continuous index process $\mathscr{X}$ at which the payoff process $\mathscr{Y}$ is even allowed to have a positive jump. When $\mathcal{P}$ collects a variety of semimartingale measures, the optimal stopping problem can be viewed as a *discretionary* stopping problem for a player who can influence both drift and volatility of the dynamic of underlying stochastic flow.

We utilize a martingale approach to construct an optimal pair $(\mathbb{P}_*,\gamma_*)$ for $\sup_{(\mathbb{P},\gamma)\in\mathcal{P}\times\mathcal{T}} \mathbb{E}_{\mathbb{P}}\big[\mathscr{Y}_{\gamma\wedge\tau_0}\big]$, in which γ_* is the first time $\mathscr{Y}$ meets the limit $\mathscr{Z}$ of its approximating $\overline{\mathscr{E}}-$Snell envelopes. To overcome the technical subtleties caused by the mutual singularity of probabilities in $\mathcal{P}$ and the discontinuity of the payoff process $\mathscr{Y}$, we approximate τ_0 by an increasing sequence of Lipschitz continuous stopping times and approximate $\mathscr{Y}$ by a sequence of uniformly continuous processes.

Keywords: discretionary stopping, random maturity, controls in weak formulation, optimal stopping, nonlinear expectation, weak stability under pasting, Lipschitz continuous stopping time, dynamic programming principle, martingale approach.

1 Introduction

We solve a continuous-time optimal stopping problem with random maturity τ_0 under an nonlinear expectation $\overline{\mathscr{E}}_0[\cdot] := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}[\cdot]$, where $\mathcal{P}$ is a weakly compact set of mutually singular probabilities on the canonical space Ω of continuous paths. More precisely, letting $\mathcal{T}$ collect all stopping times with respect to the natural filtration $\mathbf{F}$ of the canonical process B on Ω, we construct in Theorem 3.1 an optimal pair $(\mathbb{P}_*,\gamma_*)\in\mathcal{P}\times\mathcal{T}$ such that

$$\sup_{(\mathbb{P},\gamma)\in\mathcal{P}\times\mathcal{T}} \mathbb{E}_{\mathbb{P}}[\mathscr{Y}_{\gamma\wedge\tau_0}] = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}[\mathscr{Y}_{\gamma_*\wedge\tau_0}] = \mathbb{E}_{\mathbb{P}_*}\big[\mathscr{Y}_{\gamma_*\wedge\tau_0}\big]. \tag{1.1}$$

Here the payoff process takes form of $\mathscr{Y}_t := \mathbf{1}_{\{t<\tau_0\}}L_t + \mathbf{1}_{\{t\ge\tau_0\}}U_t$, $t\in[0,T]$ for two bounded processes $L\le U$ that are uniformly continuous in sense of (2.2), and the random maturity τ_0 is the hitting time to level 0 of some continuous index process $\mathscr{X}$ adapted to $\mathbf{F}$. Writing (1.1) alternatively as

$$\sup_{\gamma\in\mathcal{T}} \overline{\mathscr{E}}_0[\mathscr{Y}_{\gamma\wedge\tau_0}] = \overline{\mathscr{E}}_0[\mathscr{Y}_{\gamma_*\wedge\tau_0}], \tag{1.2}$$

we see that γ_* is an optimal stopping time for the optimal stopping with random maturity τ_0 under nonlinear expectation $\overline{\mathscr{E}}_0$. When $\mathcal{P}$ collects measures under which B is a semimartingale with uniformly bounded drift and diffusion coefficients (in this case, the nonlinear expectation $\overline{\mathscr{E}}_0$ is the G-expectation in sense of Peng [39]), the

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optimal stopping problem can be viewed as a *discretionary* stopping problem for a player who can control both drift and volatility of B's dynamic.

The optimal stopping problem with random maturity under the nonlinear expectation $\overline{\mathscr{E}}_0$ was first studied by Ekren, Touzi and Zhang [19] who took the random maturity to be the first exit time H of B from some convex open domain O and considered reward processes to have positive that have positive jumps but they do not allow for jumps at H, which is the case of interest for us. Moreover, the convexity of O is a restrictive assumption for the applications we have in mind in particular for finding an optimal triplet for robust Dynkin game in [7][1]. We extend [19] in the following two ways: First, τ_0 is more general than H so that our result can be at least applied to identify an optimal triplet for robust Dynkin game. See also Example 3.1 for τ_0's that are the first exit time of B from certain non-convex domain. Second, we impose a weaker stability under pasting assumption on the probability class than the *stability under finite pasting* used in [19].

Since the seminal work [41], the martingale approach became a primary tool in optimal stopping theory (see e.g. [35], [22], Appendix D of [26]). Like [19], we will take a martingale approach with respect to the nonlinear expectation $\overline{\mathscr{E}}_0$. As probabilities in $\mathcal{P}$ are mutually singular, one can not define the conditional expectation of $\overline{\mathscr{E}}_0$, and thus the Snell envelope of payoff process $\mathscr{Y}$, in essential supremum sense. Instead, we use shifted processes and regular conditional probability distributions (see Subsection 2.1 for details) to construct the Snell envelope Ξ of $\mathscr{Y}$ with respect to pathwise-defined nonlinear expectations $\overline{\mathscr{E}}_t[\xi](\omega) := \sup\limits_{\mathbb{P}\in\mathcal{P}_t} \mathbb{E}_\mathbb{P}[\xi^{t,\omega}]$, $(t,\omega)\in[0,T]\times\Omega$. Here $\mathcal{P}_t$ is a set of probabilities on the shifted canonical space Ω^t which includes all regular conditional probability distributions stemming from $\mathcal{P}$, see (P3). In demonstrating the martingale property of Ξ with respect to the nonlinear expectations $\overline{\mathscr{E}} = \{\overline{\mathscr{E}}_t\}_{t\in[0,T]}$, we have encountered two major technical difficulties: First, no dominating probability in $\mathcal{P}$ means no bounded convergence theorem for the nonlinear expectations $\overline{\mathscr{E}}$, then one can not follow the classical approach for optimal stopping in El Karoui [22] to obtain the $\overline{\mathscr{E}}-$martingale property of Ξ. Second, the jump of payoff process $\mathscr{Y}$ at the random maturity τ_0 and the discontinuity of each $\mathscr{Y}_t$ over Ω (because of the discontinuity of τ_0) bring technical subtleties in deriving the dynamic programming principle of Ξ, a necessity for the $\overline{\mathscr{E}}-$martingale property of Ξ.

To resolve the optimization problem (1.1), we first consider the case $Y = L = U$, however, with a Lipschitz continuous stopping time $\wp$ as the random maturity. For the modified payoff process $\widehat{Y}_t := Y_{\wp\wedge t}$, $t\in[0,T]$, we construct in Theorem 4.1 an optimal pair $(\widehat{\mathbb{P}},\widehat{\nu})\in\mathcal{P}\times\mathcal{T}$ of the corresponding optimization problem

$$\sup_{(\mathbb{P},\gamma)\in\mathcal{P}\times\mathcal{T}} \mathbb{E}_\mathbb{P}\Big[\widehat{Y}_\gamma\Big] = \mathbb{E}_{\widehat{\mathbb{P}}}\Big[\widehat{Y}_{\widehat{\nu}}\Big] \tag{1.3}$$

such that $\widehat{\nu}$ is the first time $\widehat{Y}$ meets its $\overline{\mathscr{E}}-$Snell envelope Z. Using the uniform continuity of Y and the Lipschitz continuity of $\wp$, we first derive a continuity estimate (4.2) of each Z_t on Ω, which leads to a dynamic programming principle (4.4) of Z and thus a path continuity estimate (4.5) of process Z. In virtue of (4.4), we show in Proposition 4.3 that Z is an $\overline{\mathscr{E}}-$supermartingale and that Z is also an $\overline{\mathscr{E}}-$submartingale up to each approximating stopping time ν_n of $\widehat{\nu}$, the latter of which shows that for some $\mathbb{P}_n\in\mathcal{P}$

$$Z_0 = \overline{\mathscr{E}}_0[Z_{\nu_n}] \le \mathbb{E}_{\mathbb{P}_n}[Z_{\nu_n}] + 2^{-n}. \tag{1.4}$$

Up to a subsequence, $\{\mathbb{P}_n\}_{n\in\mathbb{N}}$ has a limit $\widehat{\mathbb{P}}$ in the weakly compact probability set $\mathcal{P}$. Then as $n\to\infty$ in (1.4), we can deduce $Z_0 = \mathbb{E}_{\widehat{\mathbb{P}}}[Z_{\widehat{\nu}}]$ and thus (1.3) by leveraging the continuity estimates (4.2), (4.5) of Z as well as a similar argument to the one used in the proof of [19, Theorem 3.3] that replaces ν_n's with a sequence of quasi-continuous random variables decreasing sequence to $\widehat{\nu}$.

To approximate the general payoff process $\mathscr{Y}$ in problem (1.1), we construct in Proposition 5.1 an increasing sequence $\{\wp_n\}_{n\in\mathbb{N}}$ of Lipschitz continuous stopping times that converges to τ_0 and satisfies

$$\wp_{n+1} - \wp_n \le \frac{2T}{n+3}, \quad n\in\mathbb{N}. \tag{1.5}$$

This result together with its premises, Lemma A.5 and Lemma A.6, are among the main contributions of this paper. Given $n,k\in\mathbb{N}$, connecting L and U near $\wp_n$ with lines of slope 2^k yields a uniformly continuous process $Y^{n,k}_t := L_t + \big[1\wedge(2^k(t-\wp_n)-1)^+\big](U_t - L_t)$, $t\in[0,T]$, see Lemma 5.1. Then one can apply Theorem 4.1 to $Y^{n,k}$ and

[1] The authors would like to thank Jianfeng Zhang for an instructive discussion.

Lipschitz continuous stopping time $\wp^{n,k}:=(\wp_n+2^{1-k})\wedge T$ to find a $\mathbb{P}_{n,k}\in\mathcal{P}$ such that the $\overline{\mathscr{E}}-$Snell envelope $Z^{n,k}$ of process $\widehat{Y}^{n,k}_t:=Y^{n,k}_{\wp^{n,k}\wedge t}$, $t\in[0,T]$ satisfies

$$Z^{n,k}_0=\mathbb{E}_{\mathbb{P}_{n,k}}\Big[Z^{n,k}_{\nu_{n,k}\wedge\zeta}\Big],\quad \forall\,\zeta\in\mathcal{T}, \tag{1.6}$$

where $\nu_{n,k}$ is the first time $\widehat{Y}^{n,k}$ meets $Z^{n,k}$.

Since $\widehat{Y}^{n,k}$ differs from process $\mathscr{Y}^n_t:=\lim\limits_{k\to\infty}\widehat{Y}^{n,k}_t=\mathbf{1}_{\{t\le\wp_n\}}L_t+\mathbf{1}_{\{t>\wp_n\}}U_{\wp_n}$, $\forall\,t\in[0,T]$ only over the stochastic interval $[\![\wp_n,\wp_n+2^{1-k}]\!]$ (both processes are stopped after $\wp_n+2^{1-k}$), the uniform continuity of L and U gives rise to an inequality (5.4) on how $\widehat{Z}^{n,k}$ converges to the $\overline{\mathscr{E}}-$Snell envelope $\mathscr{Z}^n$ of $\mathscr{Y}^n$ in term of 2^{1-k}. Similarly, one can deduce from (1.5) and the uniform continuity of L,U an estimate (5.5) on the distance between $\mathscr{Z}^n$ and $\mathscr{Z}^{n+1}$, which further implies that for each $(t,\omega)\in[0,T]\times\Omega$, $\{\mathscr{Z}^n_t(\omega)\}_{n\in\mathbb{N}}$ is a Cauchy sequence, and thus admits a limit $\mathscr{Z}_t(\omega)$, see (5.6). We then show in Proposition 5.3 that $\mathscr{Z}$ is an $\mathbf{F}-$adapted continuous process that is above the $\overline{\mathscr{E}}-$Snell envelope of the stopped payoff process $\mathscr{Y}^{\tau_0}$ and stays at U_{τ_0} after the maturity τ_0, so the first time γ_* when $\mathscr{Z}$ meets $\mathscr{Y}$ precedes τ_0.

To prove our main result, Theorem 3.1, we let $n<i<\ell<m$ so that the stopping time $\zeta_{i,\ell}:=\inf\big\{t\in[0,T]: Z^{\ell,\ell}_t\le L_t+1/i\big\}$ satisfies $\zeta_{i,l}\wedge\wp_n\le\nu_{m,m}\wedge\wp_n$. Applying (5.4), (5.5) and (1.6) with $(n,k,\zeta)=\big(m,m,\zeta_{i,l}\wedge\wp_n\big)$ yields

$$\mathscr{Z}_0\le Z^{m,m}_0+\overline{\varepsilon}_m\le\mathbb{E}_{\mathbb{P}_{m,m}}\Big[Z^{m,m}_{\zeta_{i,\ell}\wedge\wp_n}\Big]+\overline{\varepsilon}_m\le\mathbb{E}_{\mathbb{P}_{m,m}}\Big[Z^{\ell,\ell}_{\zeta_{i,\ell}\wedge\wp_n}\Big]+\overline{\varepsilon}_m+\overline{\varepsilon}_\ell. \tag{1.7}$$

Let $\mathbb{P}_*$ be the limit of $\{\mathbb{P}_{m,m}\}_{m\in\mathbb{N}}$ (up to a subsequence) in the weakly compact probability set $\mathcal{P}$. As $m\to\infty$ in (1.7), we can deduce $\mathscr{Z}_0\le\mathbb{E}_{\mathbb{P}_*}\Big[\mathscr{Z}^{\ell,\ell}_{\zeta_{i,\ell}\wedge\wp_n}\Big]+\overline{\varepsilon}_\ell\le\mathbb{E}_{\mathbb{P}_*}\big[\mathscr{Z}_{\zeta_{i,\ell}\wedge\wp_n}\big]+\overline{\varepsilon}_\ell$ from (5.4), (5.5), the continuity estimates (4.2), (4.5) of $Z^{\ell,\ell}$ as well as a similar argument to the one used in the proof of [19, Theorem 3.3] that approximates $\zeta_{i,\ell}$ by a decreasing sequence of quasi-continuous random variables. Then sending ℓ,i,n to ∞ leads to

$$\mathscr{Z}_0\le\mathbb{E}_{\mathbb{P}_*}\big[\mathscr{Z}_{\gamma_*}\big]=\mathbb{E}_{\mathbb{P}_*}\big[\mathscr{Y}_{\gamma_*\wedge\tau_0}\big]\le\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_\mathbb{P}[\mathscr{Y}_{\gamma_*\wedge\tau_0}]\le\sup_{(\mathbb{P},\gamma)\in\mathcal{P}\times\mathcal{T}}\mathbb{E}_\mathbb{P}[\mathscr{Y}_{\gamma\wedge\tau_0}]\le\mathscr{Z}_0, \tag{1.8}$$

thus (1.1) holds.

Among our assumptions on the probability class $\{\mathcal{P}_t\}_{t\in[0,T]}$, (P2) is a continuity condition of the shifted canonical process B^t that is uniform at each $\mathbf{F}^t-$stopping time ($\mathbf{F}^t$ denotes the natural filtration of B^t) and under each $\mathbb{P}\in\mathcal{P}_t$. This condition together with the uniform continuity of L,U implies the path continuity (4.5) of $\overline{\mathscr{E}}-$envelope of any uniformly continuous process as well as the aforementioned estimates (5.4), (5.5) about the approximating Snell envelopes $Z^{n,k}$ and $\mathscr{Z}^n$, all are crucial for the proof of Theorem 3.1. Another important assumption we impose on the probability class $\{\mathcal{P}_t\}_{t\in[0,T]}$ is the "weak stability under pasting" (P4), which is the key to the supersolution part of the dynamic programming principle (4.4) for the $\overline{\mathscr{E}}-$envelope of any uniformly continuous process. More precisely, (P4) allows us to assemble local $\varepsilon-$optimal controls of the $\overline{\mathscr{E}}-$envelope to form approximating strategies. In Example 3.3, we show that these two assumptions along with (P3) are satisfied by controls in weak formulation i.e. $\mathcal{P}$ contains all semimartingale measures under which B has uniformly bounded drift and diffusion coefficients.

Relevant Literature. The authors analyzed in [3, 4] an optimal stopping problem under a non-linear expectation $\sup\limits_{i\in\mathcal{I}}\mathcal{E}_i[\cdot]$ over a filtered probability space $\big(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}},\widetilde{\mathbf{F}}=\big\{\widetilde{\mathcal{F}}_t\big\}_{t\in[0,T]}\big)$, where $\{\mathcal{E}_i[\cdot|\mathcal{F}_t]\}_{t\in[0,T]}$ is a $\widetilde{\mathbf{F}}-$consistent (nonlinear) expectation under $\widetilde{\mathbb{P}}$ for each index $i\in\mathcal{I}$. A notable example of $\widetilde{\mathbf{F}}-$consistent expectations are the "g-expectations" introduced by [37], which represent a fairly large class of convex risk measures, thanks to [14, 38]. If $\mathcal{E}_i$'s are conditional expected values with controls, the optimal stopping problem under $\sup\limits_{i\in\mathcal{I}}\mathcal{E}_i[\cdot]$ is exactly the classic control problem with discretionary stopping, whose general existence/characterization results can be found in [17, 31, 22, 8, 24, 32, 33, 10, 15, 29] among others. (For explicit solutions to applications of such control problems with discretionary stopping, e.g. target-tracking models and computation of the upper-hedging prices of American contingent claims under constraints, please refer to the literature in [29].) See also [9, 25, 13] for the related optimal consumption-portfolio selection problem with discretionary stopping. When the nonlinear expectation becomes $\inf\limits_{i\in\mathcal{I}}\mathcal{E}_i[\cdot]$, the optimal stopping problem considered in [3, 4] transforms to the robust optimal stopping under Knightian uncertainty or the closely related controller-stopper-game, which were also extensively studied over the past few decades: [28, 30, 23, 12, 15, 40, 2, 3, 4, 5, 11, 34] and etc.

All works cited in the last paragraph assumed that the probability set $\mathcal{P}$ is dominated by a single probability or that the controller is only allowed to affect the drift. When $\mathcal{P}$ contains mutually singular probabilities or the controller can influence not only the drift but also the volatility, there has been a little progress in research due to the technical subtleties caused by the mutual singularity of $\mathcal{P}$ such as the bounded/dominated convergence theorem generally fails in this framework. Krylov [31] solved the control problem with discretionary stopping in an one-dimensional Markov model with uniformly non-degenerate diffusion, however, his approach that relies heavily on the smoothness of the (deterministic) value function does not work in the general case. In order to extend the notion of viscosity solutions to the fully nonlinear path-dependent PDEs, as developed in [20, 18], Ekren, Touzi and Zhang [19] studied the optimal stopping problem with the random maturity $\mathtt{H}$ under the nonlinear expectation $\overline{\mathscr{E}}_0$. Our paper analyzed a similar problem, however, with allow for more general forms for τ_0 as explained above.

In spite of following its technical set-up, we adopt a quite different method than [19]: To estimate the difference between $t-$time Snell envelope values along two paths $\omega,\omega'\in\Omega$ satisfying $t<\mathtt{H}(\omega)\wedge\mathtt{H}(\omega')$, i.e. $\Delta_t(\omega,\omega'):=\underset{(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t}{\sup}\mathbb{E}_\mathbb{P}\Big[\overline{Y}^{t,\omega}_\gamma\Big]-\underset{(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t}{\sup}\mathbb{E}_\mathbb{P}\Big[\overline{Y}^{t,\omega'}_\gamma\Big]$ with $\overline{Y}_s:=Y_{H\wedge s}$, $s\in[0,T]$, [19] focuses on all trajectories traveling along the straight line $\mathfrak{l}$ from $\omega'(t)$ to $\omega(t)$ over a short period $[t,t+\delta]$. Using a "stability of finite pasting" assumption on the probability class $\{\mathcal{P}_s\}_{s\in[0,T]}$ $\big($which implies (P4), see Remark 3.1 (3)$\big)$ and the assumption that $\mathcal{P}_t|_{[t,T-\delta]}\subset\mathcal{P}_{t+\delta}$, [19] shifts distributions $\mathbb{P}$ along these trajectories from time t to time $t+\delta$. As $\mathfrak{l}$ is still inside the convex open domain, the stopping time $\mathtt{H}$ can also be transferred along these trajectories with a delay of δ. Then one can use the uniform continuity of Y to estimate $\big|\Delta_t(\omega,\omega')\big|$. On the other hand, as described above, we first solve the optimal stopping problem with Lipschitz continuous random maturity $\wp$ and then approximate the hitting time τ_0 by Lipschitz continuous stopping times.

As to the robust optimal stopping problem, or the related controller-stopper-game, with respect to the set $\mathcal{P}$ of mutually singular probabilities, Nutz and Zhang [36] and Bayraktar and Yao [6], used different methods to obtain the existence of the game value and its martingale property under the nonlinear expectations $\underline{\mathscr{E}}_t[\xi](\omega):=\underset{\mathbb{P}\in\mathcal{P}_t}{\inf}\mathbb{E}_\mathbb{P}[\xi^{t,\omega}]$, $(t,\omega)\in[0,T]\times\Omega$ (see the introduction of [6] for its comparison with [36]). Such a robust optimal stopping problem are also considered by, e.g., [27] and [1] for some particular cases, (see also [6] for a summary).

Moreover, Bayraktar and Yao [7] analyzed a robust Dynkin game with respect to the set $\mathcal{P}$ of mutually singular probabilities, they show that the Dynkin game has a value and characterize its $\underline{\mathscr{E}}-$martingale property. Applying the main result of the current paper, Theorem 3.1, [7] also reaches an optimal triplet for the robust Dynkin game.

Very recently, Ekren and Zhang [21] found that our results are useful for defining the viscosity solutions of fully non-linear degenerate path dependent PDEs.

The rest of the paper is organized as follows: Section 2 introduces some notation and preliminary results such as the regular conditional probability distribution. In section 3, we state our main result on the optimal stopping problem with random maturity τ_0 under nonlinear expectation $\overline{\mathscr{E}}_0$ after we impose some assumptions on the payoff process and the classes $\{\mathcal{P}_t\}_{t\in[0,T]}$ of mutually singular probabilities. In Section 4, we first solve an auxiliary optimal stopping problem with uniformly continuous payoff process and Lipschitz continuous random maturity under the nonlinear expectation $\overline{\mathscr{E}}_0$ by exploring the properties of the corresponding $\overline{\mathscr{E}}-$Snell envelope such as dynamic programming principles it satisfies, the path regularity properties as well as the $\overline{\mathscr{E}}-$martingale characterization. In Section 5, we approximate the hitting time τ_0 of the index process $\mathscr{X}$ by Lipschitz continuous stopping times and approximate the general payoff process $\mathscr{Y}$ with discontinuity at τ_0 by uniformly continuous processes. Then we show that the convergence of the Snell envelopes of the approximating uniformly continuous processes and derive the regularity of their limit, which is necessary to prove our main result. Section 6 contains proofs of our results while the demonstration of some auxiliary statements with starred labels in these proofs are deferred to the Appendix. We also include two technical lemmata in the appendix.

2 Notation and Preliminaries

Throughout this paper, we fix $d\in\mathbb{N}$ and a time horizon $T\in(0,\infty)$. Let $t\in[0,T]$.

We set $\Omega^t:=\big\{\omega\in\mathbb{C}\big([t,T];\mathbb{R}^d\big):\omega(t)=0\big\}$ as the canonical space over period $[t,T]$. Given $\omega\in\Omega$, $\phi^\omega_t(x):=\sup\big\{|\omega(r')-\omega(r)|:r,r'\in[0,t],\ 0\le|r'-r|\le x\big\}$, $x\in[0,t]$ is clearly a modulus of continuity function satisfying

$\lim\limits_{x\to 0+}\downarrow \phi^\omega_t(x)=0$. For any $s\in[t,T]$, $\|\omega\|_{t,s}:=\sup\limits_{r\in[t,s]}|\omega(r)|$, $\forall\,\omega\in\Omega^t$ defines a semi-norm on Ω^t. In particular, $\|\cdot\|_{t,T}$ is the *uniform* norm on Ω^t, under which Ω^t is a separable complete metric space.

The canonical process B^t of Ω^t is a $d-$dimensional standard Brownian motion under the Wiener measure $\mathbb{P}^t_0$ of $\big(\Omega^t,\mathcal{F}^t_T\big)$. Let $\mathbf{F}^t=\{\mathcal{F}^t_s\}_{s\in[t,T]}$, with $\mathcal{F}^t_s:=\sigma\big(B^t_r; r\in[t,s]\big)$, be the natural filtration of B^t and let $\mathcal{T}^t$ collect all $\mathbf{F}^t-$stopping times. Also, let $\mathfrak{P}_t$ collect all probabilities on $\big(\Omega^t,\mathcal{F}^t_T\big)$. For any $\mathbb{P}\in\mathfrak{P}_t$ and any sub-sigma-field $\mathcal{G}$ of $\mathcal{F}^t_T$, we denote by $L^1(\mathcal{G},\mathbb{P})$ the space of all real-valued, $\mathcal{G}-$measurable random variables ξ with $\|\xi\|_{L^1(\mathcal{G},\mathbb{P})}:=\mathbb{E}_\mathbb{P}\big[|\xi|\big]<\infty$.

Given $s\in[t,T]$, we set $\mathcal{T}^t_s:=\{\tau\in\mathcal{T}^t:\tau(\omega)\ge s,\ \forall\,\omega\in\Omega^t\}$ and define the *truncation* mapping Π^t_s from Ω^t to Ω^s by $\big(\Pi^t_s(\omega)\big)(r):=\omega(r)-\omega(s)$, $\forall\,(r,\omega)\in[s,T]\times\Omega^t$. By Lemma A.1 of [6], $\tau(\Pi^t_s)=\tau\circ\Pi^t_s\in\mathcal{T}^t_s$, $\forall\,\tau\in\mathcal{T}^s$. For any $\delta>0$ and $\omega\in\Omega^t$,

$$O^s_\delta(\omega):=\big\{\omega'\in\Omega^t:\|\omega'-\omega\|_{t,s}<\delta\big\}\text{ is an }\mathcal{F}^t_s-\text{measurable open set of }\Omega^t, \tag{2.1}$$

and $\overline{O}^s_\delta(\omega):=\big\{\omega'\in\Omega^t:\|\omega'-\omega\|_{t,s}\le\delta\big\}$ is an $\mathcal{F}^t_s-$measurable closed set of Ω^t $\big($see e.g. (2.1) of [6]$\big)$. In particular, we will simply denote $O^T_\delta(\omega)$ and $\overline{O}^T_\delta(\omega)$ by $O_\delta(\omega)$ and $\overline{O}_\delta(\omega)$ respectively.

We will drop the superscript t from the above notations if it is 0. For example, $(\Omega,\mathcal{F})=(\Omega^0,\mathcal{F}^0)$.

We say that ξ is a continuous random variable on Ω if for any $\omega\in\Omega$ and $\varepsilon>0$, there exists a $\delta=\delta(\omega,\varepsilon)>0$ such that $|\xi(\omega')-\xi(\omega)|<\varepsilon$ for any $\omega'\in O_\delta(\omega)$. Also, ξ is called a Lipschitz continuous random variable on Ω if for some $\kappa>0$, $|\xi(\omega')-\xi(\omega)|\le\kappa\|\omega'-\omega\|_{0,T}$ holds for any $\omega,\omega'\in\Omega$.

We say that a process X is bounded by some $C>0$ if $|X_t(\omega)|\le C$ for any $(t,\omega)\in[0,T]\times\Omega$. Also, a real-valued process X is called to be uniformly continuous on $[0,T]\times\Omega$ with respect to some modulus of continuity function ρ if

$$|X_{t_1}(\omega_1)-X_{t_2}(\omega_2)|\le\rho\Big(\mathbf{d}_\infty\big((t_1,\omega_1),(t_2,\omega_2)\big)\Big),\quad \forall\,(t_1,\omega_1),(t_2,\omega_2)\in[0,T]\times\Omega, \tag{2.2}$$

where $\mathbf{d}_\infty\big((t_1,\omega_1),(t_2,\omega_2)\big):=|t_1-t_2|+\|\omega_1(\cdot\wedge t_1)-\omega_2(\cdot\wedge t_2)\|_{0,T}$. For any $t\in[0,T]$, taking $t_1=t_2=t$ in (2.2) shows that $\big|X_t(\omega_1)-X_t(\omega_2)\big|\le\rho\big(\|\omega_1-\omega_2\|_{0,t}\big)$, $\omega_1,\omega_2\in\Omega$, which implies the $\mathcal{F}_t-$measurability of X_t. So

$$X\text{ is indeed an }\mathbf{F}-\text{adapted process with all continuous paths.} \tag{2.3}$$

Moreover, let $\mathfrak{M}$ denote all modulus of continuity functions ρ such that for some $C>0$ and $0<p_1\le p_2$,

$$\rho(x)\le C(x^{p_1}\vee x^{p_2}),\quad \forall\,x\in[0,\infty). \tag{2.4}$$

In this paper, we will frequently use the convention $\inf\emptyset:=\infty$ as well as the inequalities

$$|x\wedge a-y\wedge a|\le|x-y|\quad\text{and}\quad|x\vee a-y\vee a|\le|x-y|,\quad\forall\,a,x,y\in\mathbb{R}. \tag{2.5}$$

2.1 Shifted Processes and Regular Conditional Probability Distributions

In this subsection, we fix $0\le t\le s\le T$. The concatenation $\omega\otimes_s\widetilde{\omega}$ of an $\omega\in\Omega^t$ and an $\widetilde{\omega}\in\Omega^s$ at time s:

$$\big(\omega\otimes_s\widetilde{\omega}\big)(r):=\omega(r)\,\mathbf{1}_{\{r\in[t,s)\}}+\big(\omega(s)+\widetilde{\omega}(r)\big)\,\mathbf{1}_{\{r\in[s,T]\}},\quad\forall\,r\in[t,T]$$

defines another path in Ω^t. Set $\omega\otimes_s\emptyset=\emptyset$ and $\omega\otimes_s\widetilde{A}:=\big\{\omega\otimes_s\widetilde{\omega}:\widetilde{\omega}\in\widetilde{A}\big\}$ for any non-empty subset $\widetilde{A}$ of Ω^s.

Lemma 2.1. *If $A\in\mathcal{F}^t_s$, then $\omega\otimes_s\Omega^s\subset A$ for any $\omega\in A$.*

For any $\mathcal{F}^t_s-$measurable random variable η, since $\{\omega'\in\Omega^t:\eta(\omega')=\eta(\omega)\}\in\mathcal{F}^t_s$, Lemma 2.1 implies that

$$\omega\otimes_s\Omega^s\subset\{\omega'\in\Omega^t:\eta(\omega')=\eta(\omega)\}\quad\text{i.e.,}\quad\eta(\omega\otimes_s\widetilde{\omega})=\eta(\omega),\quad\forall\,\widetilde{\omega}\in\Omega^s. \tag{2.6}$$

To wit, the value $\eta(\omega)$ depends only on $\omega|_{[t,s]}$.

Let $\omega\in\Omega^t$. For any $A\subset\Omega^t$ we set $A^{s,\omega}:=\{\widetilde{\omega}\in\Omega^s:\omega\otimes_s\widetilde{\omega}\in A\}$ as the projection of A on Ω^s along ω. In particular, $\emptyset^{s,\omega}=\emptyset$. Given a random variable ξ on Ω^t, define the *shift* $\xi^{s,\omega}$ of ξ along $\omega|_{[t,s]}$ by $\xi^{s,\omega}(\widetilde{\omega}):=\xi(\omega\otimes_s\widetilde{\omega})$, $\forall\,\widetilde{\omega}\in\Omega^s$. Correspondingly, for a process $X=\{X_r\}_{r\in[t,T]}$ on Ω^t, its *shifted* process $X^{s,\omega}$ is

$$X^{s,\omega}(r,\widetilde{\omega}):=(X_r)^{s,\omega}(\widetilde{\omega})=X_r(\omega\otimes_s\widetilde{\omega}),\quad\forall\,(r,\widetilde{\omega})\in[s,T]\times\Omega^s.$$

Shifted random variables and shifted processes "inherit" the measurability of original ones:

Proposition 2.1. *Let $0 \le t \le s \le T$ and $\omega \in \Omega^t$.*
(1) If a real-valued random variable ξ on Ω^t is $\mathcal{F}^t_r-$measurable for some $r \in [s,T]$, then $\xi^{s,\omega}$ is $\mathcal{F}^s_r-$measurable.
(2) For any $\tau \in \mathcal{T}^t$, if $\tau(\omega \otimes_s \Omega^s) \subset [r,T]$ for some $r \in [s,T]$, then $\tau^{s,\omega} \in \mathcal{T}^s_r$.
(3) If a real-valued process $\{X_r\}_{r\in[t,T]}$ is $\mathbf{F}^t-$adapted (resp. $\mathbf{F}^t-$progressively measurable), then $X^{s,\omega}$ is $\mathbf{F}^s-$adapted (resp. $\mathbf{F}^s-$progressively measurable).

Let $\mathbb{P} \in \mathfrak{P}_t$. In light of the *regular conditional probability distributions* (see e.g. [43]), we can follow Section 2.2 of [6] to introduce a family of *shifted* probabilities $\{\mathbb{P}^{s,\omega}\}_{\omega\in\Omega^t} \subset \mathfrak{P}_s$, under which the corresponding shifted random variables inherit the $\mathbb{P}$ integrability of original ones:

Proposition 2.2. *If $\xi \in L^1\big(\mathcal{F}^t_T, \mathbb{P}\big)$ for some $\mathbb{P} \in \mathfrak{P}_t$, then it holds for $\mathbb{P}-$a.s. $\omega \in \Omega^t$ that $\xi^{s,\omega} \in L^1\big(\mathcal{F}^s_T, \mathbb{P}^{s,\omega}\big)$ and*

$$\mathbb{E}_{\mathbb{P}^{s,\omega}}\big[\xi^{s,\omega}\big] = \mathbb{E}_{\mathbb{P}}\big[\xi\big|\mathcal{F}^t_s\big](\omega) \in \mathbb{R}. \tag{2.7}$$

This subsection was presented in [6] with more details and proofs.

3 Main Results

In this section, after imposing some assumptions on the payoff process and the classes $\{\mathcal{P}_t\}_{t\in[0,T]}$ of mutually singular probabilities, we will present our main result, Theorem 3.1, on the optimal stopping problem under the nonlinear expectation $\overline{\mathscr{E}}_0[\cdot] := \sup\limits_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}[\cdot]$, whose random maturity is in form of the hitting time τ_0 to level 0 of some continuous index process $\mathscr{X}$. More precisely, let $\mathscr{X}$ be a process with $\mathscr{X}_0 > 0$ such that all its paths are continuous and that for some modulus of continuity function $\rho_{\mathscr{X}}$

$$|\mathscr{X}_t(\omega) - \mathscr{X}_t(\omega')| \le \rho_{\mathscr{X}}(\|\omega - \omega'\|_{0,t}), \quad \forall\, t \in [0,T], \ \forall\, \omega, \omega' \in \Omega. \tag{3.1}$$

Clearly, (3.1) implies that the $\mathbf{F}-$adaptedness of $\mathscr{X}$. Then $\tau_0 := \inf\{t \in [0,T] : \mathscr{X}_t \le 0\} \wedge T \in (0,T]$ is an $\mathbf{F}-$stopping time and

$$\tau_n := \inf\big\{t \in [0,T] : \mathscr{X}_t \le \big(\lceil \log_2(n+2)\rceil + \lfloor \mathscr{X}_0^{-1} \rfloor - 1\big)^{-1}\big\} \wedge T \in (0, \tau_0], \quad n \in \mathbb{N} \tag{3.2}$$

is an increasing sequence of $\mathbf{F}-$stopping times that converges to τ_0.

The following example shows that τ_0 could be the first exit time of B from some non-convex domain.

Example 3.1. *1) Let $d = 2$. Clearly, $\mathscr{X}_t = 1 + B^{(2)}_t + \big|B^{(1)}_t\big|$, $t \in [0,T]$ defines a process with $\mathscr{X}_0 = 1$ such that all its paths are continuous and that for any $t \in [0,T]$ and $\omega, \omega' \in \Omega$, $|\mathscr{X}_t(\omega) - \mathscr{X}_t(\omega')| \le \big|B^{(1)}_t(\omega) - B^{(1)}_t(\omega')\big| + \big|B^{(2)}_t(\omega) - B^{(2)}_t(\omega')\big| \le 2|B_t(\omega) - B_t(\omega')| \le 2\|\omega - \omega'\|_{0,t}$. However, $\tau_0 = \inf\{t \in [0,T] : \mathscr{X}_t \le 0\} \wedge T = \inf\{t \in [0,T] : B_t \notin \Upsilon\} \wedge T$ is the first exit time of B from $\Upsilon := \{(x,y) \in \mathbb{R}^2 : y > -1 - |x|\}$, a non-convex subset of $\mathbb{R}^2$.*
2) Let $d = 2$ and let $\Gamma := \{(r\cos\theta, r\sin\theta) : r \in [0,1],\ \theta \in [0, \frac{3}{2}\pi]\}$ be the 3/4 unit disk in $\mathbb{R}^2$ centered at the origin $(0,0)$. Clearly, $\mathscr{X}_t := 1/2 - dist(B_t, \Gamma)$, $t \in [0,T]$ is a process with $\mathscr{X}_0 = 1/2$ such that all its paths are continuous and that for any $t \in [0,T]$ and $\omega, \omega' \in \Omega$, $|\mathscr{X}_t(\omega) - \mathscr{X}_t(\omega')| \le |dist(B_t(\omega), \Gamma) - dist(B_t(\omega'), \Gamma)| \le |B_t(\omega) - B_t(\omega')| \le \|\omega - \omega'\|_{0,t}$. However, $\tau_0 = \inf\{t \in [0,T] : \mathscr{X}_t \le 0\} \wedge T = \inf\{t \in [0,T] : B_t \notin \widetilde{\Gamma}\} \wedge T$ is the first exit time of B from $\widetilde{\Gamma} := \{(x,y) \in \mathbb{R}^2 : dist((x,y), \Gamma) < 1/2\}$, another non-convex subset of $\mathbb{R}^2$.

3.1 Uniform Continuity of Payoff Processes

Standing assumptions on payoff processes (L, U).

Let L and U be two real-valued processes bounded by some $M_0 > 0$ such that
(A1) both are uniformly continuous on $[0,T] \times \Omega$ with respect to some $\rho_0 \in \mathfrak{M}$ such that ρ_0 satisfies (2.4) with some $\mathfrak{C} > 0$ and $0 < \mathfrak{p}_1 \le \mathfrak{p}_2$;
(A2) $L_t(\omega) \le U_t(\omega)$, $\forall\, (t,\omega) \in [0,T) \times \Omega$ and $L_T(\omega) = U_T(\omega)$, $\forall\, \omega \in \Omega$.

We consider the following payoff process

$$\mathscr{Y}_t := \mathbf{1}_{\{t<\tau_0\}} L_t + \mathbf{1}_{\{t\ge\tau_0\}} U_t, \quad t \in [0,T], \tag{3.3}$$

Clearly, $\mathscr{Y}$ is an $\mathbf{F}-$adapted process bounded by M_0 whose paths are all continuous except a possible positive jump at τ_0.

Example 3.2. *1) (American-type contingent claims for controllers) Consider an American-type contingent claim for an agent who is able to influence the probability model via certain controls (e.g. an insider): The claim pays the agent an endowment $U_{\tau_{\mathfrak{J}}}$ at the first time $\tau_{\mathfrak{J}}$ when some financial index process $\mathfrak{J}$ rises to certain level $\mathfrak{a}$ (Taking $\mathscr{X}_t=\mathfrak{a}-\mathfrak{J}_t$, $t\in[0,T]$ shows that $\tau_0=\tau_{\mathfrak{J}}$). If the agent chooses to exercise at an earlier time γ than $\tau_{\mathfrak{J}}$, she will receive L_γ. Then the price of such an American-type contingent claim is $\sup\limits_{(\mathbb{P},\gamma)\in\mathcal{P}\times\mathcal{T}}\mathbb{E}_{\mathbb{P}}[\mathscr{Y}_{\gamma\wedge\tau_0}]$.*

2) (robust Dynkin game) [7] analyzed a robust Dynkin game with respect to the set $\mathcal{P}$ of mutually singular probabilities: Player 1 (who conservatively thinks that the Nature is not in favor of her) will receive from Player 2 a payoff $R(\tau,\gamma):=\mathbf{1}_{\{\tau\le\gamma\}}\mathfrak{L}_\tau+\mathbf{1}_{\{\gamma<\tau\}}\mathfrak{U}_\gamma$ if they choose to exit the game at $\tau\in\mathcal{T}$ and $\gamma\in\mathcal{T}$ respectively. The paper shows that Player 1 has a value in the robust Dynkin game, i.e. $V=\inf\limits_{\mathbb{P}\in\mathcal{P}}\inf\limits_{\gamma\in\mathcal{T}}\sup\limits_{\tau\in\mathcal{T}}\mathbb{E}_{\mathbb{P}}\big[R(\tau,\gamma)\big]=\sup\limits_{\tau\in\mathcal{T}}\inf\limits_{\gamma\in\mathcal{T}}\inf\limits_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_{\mathbb{P}}\big[R(\tau,\gamma)\big]$ and identifies an optimal stopping time τ_* for Player 1, which is the first time Player 1's value process meets $\mathfrak{L}$ (see Theorem 5.1 therein). Then the robust Dynkin game reduces to the optimal stopping problem with random maturity τ_* under nonlinear expectation $\overline{\mathscr{E}}_0:=\sup\limits_{\mathbb{P}\in\mathcal{P}}\mathbb{E}_{\mathbb{P}}[\ ]$, i.e. $\sup\limits_{(\mathbb{P},\gamma)\in\mathcal{P}\times\mathcal{T}}\mathbb{E}_{\mathbb{P}}[\mathscr{Y}_{\gamma\wedge\tau_*}]$, where $L:=-\mathfrak{U}$ and $U:=-\mathfrak{L}$.

3.2 Weak Stability under Pasting

Let $\mathfrak{S}$ collect all pairs $(Y,\wp)$ such that
(i) Y is a real-valued process bounded by $M_Y>0$ and uniformly continuous on $[0,T]\times\Omega$ with respect to some $\rho_Y\in\mathfrak{M}$;
(ii) $\wp\in\mathcal{T}$ is a Lipschitz continuous stopping time on Ω with coefficient $\kappa_\wp>0$: $|\wp(\omega)-\wp(\omega')|\le\kappa_\wp\|\omega-\omega'\|_{0,T}$, $\forall\,\omega,\omega'\in\Omega$.

For any $(Y,\wp)\in\mathfrak{S}$, we define

$$\widehat{Y}_t:=Y_{\wp\wedge t},\quad t\in[0,T],\tag{3.4}$$

which is clearly an $\mathbf{F}-$adapted process bounded by M_Y that has all continuous paths.

Standing assumptions on probability class.

We consider a family $\{\mathcal{P}_t\}_{t\in[0,T]}$ of subsets of $\mathfrak{P}_t$, $t\in[0,T]$ such that
(**P1**) $\mathcal{P}:=\mathcal{P}_0$ is a weakly compact subset of $\mathfrak{P}_0$.
(**P2**) For any $\rho\in\mathfrak{M}$, there exists another $\widehat{\rho}$ of $\mathfrak{M}$ such that

$$\sup_{(\mathbb{P},\zeta)\in\mathcal{P}_t\times\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\bigg[\rho\Big(\delta+\sup_{r\in[\zeta,(\zeta+\delta)\wedge T]}\big|B^t_r-B^t_\zeta\big|\Big)\bigg]\le\widehat{\rho}(\delta),\quad\forall\,t\in[0,T),\ \forall\,\delta\in(0,\infty).\tag{3.5}$$

In particular, we require $\widehat{\rho}_0$ to satisfy (2.4) with some $\widehat{\mathfrak{C}}>0$ and $1<\widehat{\mathfrak{p}}_1\le\widehat{\mathfrak{p}}_2$.
(**P3**) For any $0\le t<s\le T$, $\omega\in\Omega$ and $\mathbb{P}\in\mathcal{P}_t$, there exists an extension $(\Omega^t,\mathcal{F}',\mathbb{P}')$ of $(\Omega^t,\mathcal{F}^t_T,\mathbb{P})$ $\big($i.e. $\mathcal{F}^t_T\subset\mathcal{F}'$ and $\mathbb{P}'|_{\mathcal{F}^t_T}=\mathbb{P}\big)$ and $\Omega'\in\mathcal{F}'$ with $\mathbb{P}'(\Omega')=1$ such that $\mathbb{P}^{s,\widetilde{\omega}}$ belongs to $\mathcal{P}_s$ for any $\widetilde{\omega}\in\Omega'$.
(**P4**) For any $(Y,\wp)\in\mathfrak{S}$, there exists a modulus of continuity function $\overline{\rho}_Y$ such that the following statement holds for any $0\le t<s\le T$, $\omega\in\Omega$ and $\mathbb{P}\in\mathcal{P}_t$: Given $\delta\in\mathbb{Q}_+$ and $\lambda\in\mathbb{N}$, let $\{\mathcal{A}_j\}_{j=0}^\lambda$ be a $\mathcal{F}^t_s-$partition of Ω^t such that for $j=1,\cdots,\lambda$, $\mathcal{A}_j\subset O^s_{\delta_j}(\widetilde{\omega}_j)$ for some $\delta_j\in\big((0,\delta]\cap\mathbb{Q}\big)\cup\{\delta\}$ and $\widetilde{\omega}_j\in\Omega^t$. Then for any $\{\mathbb{P}_j\}_{j=1}^\lambda\subset\mathcal{P}_s$, there is a $\widehat{\mathbb{P}}=\widehat{\mathbb{P}}(Y,\wp)\in\mathcal{P}_t$ such that
(i) $\widehat{\mathbb{P}}(A\cap\mathcal{A}_0)=\mathbb{P}(A\cap\mathcal{A}_0)$, $\forall\,A\in\mathcal{F}^t_T$;
(ii) For any $j=1,\cdots,\lambda$ and $A\in\mathcal{F}^t_s$, $\widehat{\mathbb{P}}(A\cap\mathcal{A}_j)=\mathbb{P}(A\cap\mathcal{A}_j)$ and

$$\mathbb{E}_{\widehat{\mathbb{P}}}\Big[\mathbf{1}_{A\cap\mathcal{A}_j}\widehat{Y}^{t,\omega}_{\gamma(\Pi^t_s)}\Big]\ge\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}_j\}}\Big(\mathbb{E}_{\mathbb{P}_j}\big[\widehat{Y}^{s,\omega\otimes_t\widetilde{\omega}}_\gamma\big]-\overline{\rho}_Y(\delta)\Big)\Big],\quad\forall\,\gamma\in\mathcal{T}^s.\tag{3.6}$$

What follows is the main result of this paper on the solvability of the optimization problem (1.1).

Theorem 3.1. *Assume* (3.1)*, (A1), (A2) and (P1)$-$(P4). Then the optimization problem* (1.1) *admits an optimal pair $(\mathbb{P}_*,\gamma_*)\in\mathcal{P}\times\mathcal{T}$, where the form of γ_* will be specified in Proposition 5.3 (4).*

For any $\mathcal{F}_T-$measurable random variable ξ that is bounded by some $C>0$, we define its nonlinear expectations with respect to the probability class $\{\mathcal{P}_t\}_{t\in[0,T]}$ by

$$\overline{\mathscr{E}}_t[\xi](\omega):=\sup_{\mathbb{P}\in\mathcal{P}_t}\mathbb{E}_{\mathbb{P}}[\xi^{t,\omega}],\quad\forall\,(t,\omega)\in[0,T]\times\Omega.$$

Then (1.1) can be alternatively expressed as (1.2), namely, γ_* alone is an optimal stopping time for the optimal stopping under nonlinear expectation $\overline{\mathscr{E}}_0$.

Remark 3.1. *(1) Clearly, $\varrho(x):=x$, $\forall\, x\in[0,\infty)$ is a modulus of continuity function in $\mathfrak{M}$. Let $\widehat{\varrho}$ be its corresponding element in $\mathfrak{M}$ in (P2) and assume that $\widehat{\varrho}$ satisfies (2.4) for some $C_\varrho>0$ and $0<q_1\le q_2$.*
(2) Based on (P2), the expectation on the right-hand-side of (3.6) is well-defined since the mapping $\widetilde{\omega}\to\mathbb{E}_{\widetilde{\mathbb{P}}}\Big[\widehat{Y}^{s,\omega\otimes_t\widetilde{\omega}}_\gamma\Big]$ is continuous under norm $\|\ \|_{t,T}$ for any $\widetilde{\mathbb{P}}\in\mathfrak{P}_s$ and $\gamma\in\mathcal{T}^s$.
(3) Analogous to the assumption (P2) of [6], the condition (P4) can be regarded as a weak form of stability under pasting since it is implied by the "stability under finite pasting" $\big($see e.g. (4.18) of [42]$\big)$: for any $0\le t<s\le T$, $\omega\in\Omega$, $\mathbb{P}\in\mathcal{P}_t$, $\delta\in\mathbb{Q}_+$ and $\lambda\in\mathbb{N}$, let $\{\mathcal{A}_j\}_{j=0}^{\lambda}$ be a $\mathcal{F}^t_s-$partition of Ω^t such that for $j=1,\cdots,\lambda$, $\mathcal{A}_j\subset O^s_{\delta_j}(\widetilde{\omega}_j)$ for some $\delta_j\in\big((0,\delta]\cap\mathbb{Q}\big)\cup\{\delta\}$ and $\widetilde{\omega}_j\in\Omega^t$. Then for any $\{\mathbb{P}_j\}_{j=1}^{\lambda}\subset\mathcal{P}_s$, the probability defined by

$$\widehat{\mathbb{P}}(A)=\mathbb{P}(A\cap\mathcal{A}_0)+\sum_{j=1}^{\lambda}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\widetilde{\omega}\in\mathcal{A}_j\}}\mathbb{P}_j\big(A^{s,\widetilde{\omega}}\big)\Big],\quad \forall\, A\in\mathcal{F}^t_T \tag{3.7}$$

is in $\mathcal{P}_t$.

Example 3.3. *(Controls of weak formulation) Given $\ell>0$, let $\{\mathcal{P}^\ell_t\}_{t\in[0,T]}$ be the family of semimartingale measures considered in [19] such that $\mathcal{P}^\ell_t$ collects all continuous semimartingale measures on $(\Omega^t,\mathcal{F}^t_T)$, whose drift and diffusion characteristics are bounded by ℓ and $\sqrt{2\ell}$ respectively. According to Lemma 2.3 therein, $\{\mathcal{P}^\ell_t\}_{t\in[0,T]}$ satisfies (P1), (P3) and stability under finite pasting $\big($thus (P4) by Remark 3.1 (3)$\big)$. Also, one can deduce from the Burkholder-Davis-Gundy inequality that $\{\mathcal{P}^\ell_t\}_{t\in[0,T]}$ satisfies (P2), see Section 6 for details.*

4 Optimal Stopping with Lipschitz Continuous Random Maturity

To solve the optimization problem (1.1) we first analyze in this section an auxiliary optimal stopping problem with uniformly continuous payoff process and Lipschitz continuous random maturity under the nonlinear expectation $\overline{\mathscr{E}}_0$.

Let the probability class $\{\mathcal{P}_t\}_{t\in[0,T]}$ satisfy (P2)$-$(P4). To solve (1.1), we first consider the case $Y=L=U$ with random maturity $\wp$ for some $(Y,\wp)\in\mathfrak{S}$. For any $(t,\omega)\in[0,T]\times\Omega$, define

$$Z_t(\omega):=\sup_{(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\Big[\widehat{Y}^{t,\omega}_\gamma\Big]$$

as a Snell envelope of the payoff process $\widehat{Y}$ with respect to the nonlinear expectations $\overline{\mathscr{E}}=\{\overline{\mathscr{E}}_t\}_{t\in[0,T]}$ given the historical path $\omega|_{[0,t]}$. We will simply refer to Z as the $\overline{\mathscr{E}}-$Snell envelope of $\widehat{Y}$. Since the $\mathbf{F}-$adaptedness of $\widehat{Y}$ and (2.6) imply that $\widehat{Y}^{t,\omega}_t(\widetilde{\omega})=\widehat{Y}_t(\omega\otimes_t\widetilde{\omega})=\widehat{Y}_t(\omega)$, $\forall\,\widetilde{\omega}\in\Omega^t$, one has

$$M_Y\ge Z_t(\omega)\ge\sup_{\mathbb{P}\in\mathcal{P}_t}\mathbb{E}_{\mathbb{P}}\Big[\widehat{Y}^{t,\omega}_t\Big]=\widehat{Y}_t(\omega)\ge -M_Y,\quad \forall\,(t,\omega)\in[0,T]\times\Omega. \tag{4.1}$$

Given $t\in[0,T]$, we have the following estimate on the continuity of random variable Z_t at each $\omega\in\Omega$, which is not only in term of the distance from ω under $\|\ \|_{0,t}$ but also in term of the path information of ω up to time t.

Proposition 4.1. *Assume (P2). Let $(Y,\wp)\in\mathfrak{S}$ and $(t,\omega)\in[0,T]\times\Omega$. It holds for any $\omega'\in\Omega$*

$$|Z_t(\omega)-Z_t(\omega')|\le\widehat{\rho}_Y\Big((1+\kappa_\wp)\|\omega-\omega'\|_{0,t}+\sup_{r\in[t_1,t_2]}\big|\omega(r)-\omega(t_1)\big|\Big)\le\widehat{\rho}_Y\Big((1+\kappa_\wp)\|\omega-\omega'\|_{0,t}+\phi^\omega_t\big(\kappa_\wp\|\omega-\omega'\|_{0,t}\big)\Big), \tag{4.2}$$

where $t_1:=\wp(\omega)\wedge\wp(\omega')\wedge t$ and $t_2:=\big(\wp(\omega)\vee\wp(\omega')\big)\wedge t$. Consequently, $Z_t(\omega)$ is continuous in ω under the norm $\|\ \|_{0,t}$: i.e. for any $\varepsilon>0$, there exists a $\delta=\delta(t,\omega)>0$ such that

$$|Z_t(\omega')-Z_t(\omega)|<\varepsilon,\quad \forall\,\omega'\in O^t_\delta(\omega), \tag{4.3}$$

and thus Z_t is $\mathcal{F}_t-$measurable.

The resolution of the auxiliary optimization problem (1.3) with the payoff process Y and random maturity $\wp$ relies on the following dynamic programming principle for the $\overline{\mathscr{E}}-$Snell envelope Z of $\widehat{Y}$ and a consequence of it, a path continuity estimate of process Z:

Proposition 4.2. *Assume (P2)−(P4). Let $(Y,\wp)\in\mathfrak{S}$. It holds for any $(t,\omega)\in[0,T]\times\Omega$ and $\nu\in\mathcal{T}^t$ that*

$$Z_t(\omega)=\sup_{(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\gamma<\nu\}}\widehat{Y}^{t,\omega}_{\gamma}+\mathbf{1}_{\{\gamma\ge\nu\}}Z^{t,\omega}_{\nu}\Big]. \tag{4.4}$$

Consequently, Z is an $\mathbf{F}-$adapted process bounded by M_Y that has all continuous paths. More precisely, for any $\omega\in\Omega$ and $0\le t\le s\le T$,

$$\big|Z_t(\omega)-Z_s(\omega)\big|\le 2C_\varrho M_Y\Big((s-t)^{\frac{q_1}{2}}\vee(s-t)^{q_2-\frac{q_1}{2}}\Big)+\widehat{\rho}_Y(s-t)+\widehat{\rho}_Y\big(\delta_{t,s}(\omega)\big)\vee\widehat{\widehat{\rho}}_Y\big(\delta_{t,s}(\omega)\big), \tag{4.5}$$

where $\delta_{t,s}(\omega):=(1+\kappa_\wp)\Big((s-t)^{\frac{q_1}{2}}+\sup_{t\le r<r'\le s}\big|\omega(r')-\omega(r)\big|\Big)$ and $\widehat{\widehat{\rho}}_Y\in\mathfrak{M}$ is the modulus of continuity function corresponding to $\widehat{\rho}_Y$ in (P2). See Remark 3.1 (1) for the notations C_ϱ,q_1 and q_2 here.

In light of Proposition 4.2, the $\overline{\mathscr{E}}-$Snell envelope Z of $\widehat{Y}$ has the following $\overline{\mathscr{E}}-$martingale properties:

Proposition 4.3. *Assume (P2)−(P4). Let $(Y,\wp)\in\mathfrak{S}$ and $n\in\mathbb{N}$. Then Z is an $\overline{\mathscr{E}}-$supermartingale, and Z is an $\overline{\mathscr{E}}-$submartingale over $[0,\nu_n]$ in sense that for any $\zeta\in\mathcal{T}$*

$$Z_{\zeta\wedge t}(\omega)\ge\overline{\mathscr{E}}_t[Z_\zeta](\omega)\quad\text{and}\quad Z_{\nu_n\wedge\zeta\wedge t}(\omega)\le\overline{\mathscr{E}}_t[Z_{\nu_n\wedge\zeta}](\omega),\quad\forall\,(t,\omega)\in[0,T]\times\Omega,$$

where $\nu_n:=\inf\big\{t\in[0,T]:Z_t-\widehat{Y}_t\le\frac{1}{n}\big\}\in\mathcal{T}$.

Exploiting the $\overline{\mathscr{E}}-$submartingale of Z up to ν_n as well as the continuity estimates (4.2), (4.5) of Z, we can solve the optimization problem (1.3) by taking a similar argument to the one used in the proof of [19, Theorem 3.3].

Theorem 4.1. *Assume (P1)−(P4) and let $(Y,\wp)\in\mathfrak{S}$. There exists a $\widehat{\mathbb{P}}\in\mathcal{P}$ such that $Z_0=\overline{\mathscr{E}}_0\big[Z_{\widehat{\nu}}\big]=\mathbb{E}_{\widehat{\mathbb{P}}}\big[Z_{\widehat{\nu}}\big]=\mathbb{E}_{\widehat{\mathbb{P}}}\big[\widehat{Y}_{\widehat{\nu}}\big]$, where $\widehat{\nu}:=\inf\big\{t\in[0,T]:Z_t=\widehat{Y}_t\big\}\in\mathcal{T}$. To wit, $\big(\widehat{\nu},\widehat{\mathbb{P}}\big)$ solves the optimization problem (1.3) with the payoff process $\widehat{Y}$. Moreover, it holds for any $\zeta\in\mathcal{T}$ that $Z_0=\overline{\mathscr{E}}_0\big[Z_{\widehat{\nu}\wedge\zeta}\big]=\mathbb{E}_{\widehat{\mathbb{P}}}\big[Z_{\widehat{\nu}\wedge\zeta}\big]$.*

5 Optimal Stopping with Random Maturity τ_0

In this section, we approximate the hitting time τ_0 of the index process $\mathscr{X}$ by Lipschitz continuous stopping times and approximate the general payoff process $\mathscr{Y}$ in (3.3) by uniformly continuous processes. We show the convergence of the Snell envelopes of the approximating uniformly continuous processes and derive the regularity of their limit, which is crucial for the proof of our main result, Theorem 3.1.

To apply Theorem 3.1, we first approximate τ_0 by an increasing sequence $\{\wp_n\}_{n\in\mathbb{N}}$ of Lipschitz continuous stopping times such that the increment $\wp_{n+1}-\wp_n$ uniformly decreases to 0 as $n\to\infty$:

Proposition 5.1. *Assume (3.1). There exist an increasing sequence $\{\wp_n\}_{n\in\mathbb{N}}$ in $\mathcal{T}$ and an increasing sequence $\{\kappa_n\}_{n\in\mathbb{N}}$ of positive numbers with $\lim_{n\to\infty}\uparrow\kappa_n=\infty$ such that for any $n\in\mathbb{N}$*
(1) $\tau_n(\omega)\le\wp_n(\omega)\le\tau_0(\omega)$ and $0\le\wp_{n+1}(\omega)-\wp_n(\omega)\le\frac{2T}{n+3}$, $\forall\,\omega\in\Omega$. In particular, if $\{t\in[0,T]:\mathscr{X}_t(\omega')\le 0\}$ is not empty for some $\omega'\in\Omega$, then $\wp_n(\omega')<\tau_0(\omega')$.
(2) Given $\omega_1,\omega_2\in\Omega$, $\big|\wp_n(\omega_1)-\wp_n(\omega_2)\big|\le\kappa_n\|\omega_1-\omega_2\|_{0,t_0}$ holds for any $t_0\in\big\{t\in[a_n,T):t\ge a_n+\kappa_n\|\omega_1-\omega_2\|_{0,t}\big\}\cup\{T\}$, where $a_n:=\wp_n(\omega_1)\wedge\wp_n(\omega_2)$.

Let $n,k\in\mathbb{N}$ and let $\wp_n$ be the $\mathbf{F}-$stopping time stated in Proposition 5.1. We use lines of slope 2^k to connect L and U near $\wp_n$ as follows: For any $t\in[0,T]$,

$$\begin{aligned}Y^{n,k}_t&:=L_t+\big[1\wedge(2^k(t-\wp_n)-1)^+\big](U_t-L_t) &(5.1)\\ &=\mathbf{1}_{\{t\le\wp_n+2^{-k}\}}L_t+\mathbf{1}_{\{\wp_n+2^{-k}<t<\wp_n+2^{1-k}\}}\Big\{\big[1-2^k(t-\wp_n-2^{-k})\big]L_t+2^k(t-\wp_n-2^{-k})U_t\Big\}+\mathbf{1}_{\{t\ge\wp_n+2^{1-k}\}}U_t, &(5.2)\end{aligned}$$

where the set $\{\wp_n(\omega)+2^{-k}<t<\wp_n(\omega)+2^{1-k}\}$ $\big($resp. $\{t\ge\wp_n(\omega)+2^{1-k}\}\big)$ may be empty if $\wp_n(\omega)+2^{-k}\ge T$ $\big($resp. $\wp_n(\omega)+2^{1-k}>T\big)$ for some $\omega\in\Omega$.

Clearly, the process $Y^{n,k}$ is also bounded by M_0, and it is uniformly continuous on $[0,T]\times\Omega$ with respect to some $\rho_{n,k}\in\mathfrak{M}$:

Lemma 5.1. *Assume* (3.1) *and (A1). For any $n,k\in\mathbb{N}$, $Y^{n,k}$ is uniformly continuous on $[0,T]\times\Omega$ with respect to the modulus of continuity function $\rho_{n,k}(x):=6\rho_0(2x)+2^{1+k}M_0(1+\kappa_n)x\le C_{n,k}\big(x^{\mathfrak{p}_1\wedge 1}\vee x^{\mathfrak{p}_2\vee 1}\big)$, $\forall\,x\in[0,\infty)$, where $C_{n,k}:=6\cdot 2^{\mathfrak{p}_2}\mathfrak{C}+2^{1+k}M_0(1+\kappa_n)$ and $\{\kappa_n\}_{n\in\mathbb{N}}$ is the increasing sequence of positive numbers in Proposition 5.1.*

Applying Proposition 5.1 (2) with $t_0=T$ shows that $\wp_n$ is a Lipschitz continuous stopping time on Ω with coefficient κ_n, so is $\wp^{n,k}:=(\wp_n+2^{1-k})\wedge T$ by (2.5). Then we define

$$\begin{aligned}\widehat{Y}^{n,k}_t:=Y^{n,k}_{\wp^{n,k}\wedge t}=Y^{n,k}_{(\wp_n+2^{1-k})\wedge t}&=\mathbf{1}_{\{t\le\wp_n+2^{-k}\}}L_t+\mathbf{1}_{\{\wp_n+2^{-k}<t<\wp_n+2^{1-k}\}}\Big\{\big[1-2^k(t-\wp_n-2^{-k})\big]L_t+2^k(t-\wp_n-2^{-k})U_t\Big\}\\&\quad+\mathbf{1}_{\{t\ge\wp_n+2^{1-k}\}}U_{(\wp_n+2^{1-k})\wedge T},\qquad\forall\,t\in[0,T],\end{aligned}\tag{5.3}$$

and its $\overline{\mathscr{E}}-$Snell envelope:

$$Z^{n,k}_t(\omega):=\sup_{(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\Big[\big(\widehat{Y}^{n,k}\big)^{t,\omega}_\gamma\Big],\qquad\forall\,(t,\omega)\in[0,T]\times\Omega.$$

As L and U are bounded by M_0, so are $Y^{n,k}$ and $Z^{n,k}$ by (4.1). In light of Lemma 5.1, we can apply the results in Section 4 to each $Z^{n,k}$, $n,k\in\mathbb{N}$.

Given $t\in[0,T]$, (5.1) shows that $\lim\limits_{k\to\infty}\uparrow Y^{n,k}_t=\mathbf{1}_{\{t\le\wp_n\}}L_t+\mathbf{1}_{\{t>\wp_n\}}U_t$. Since

$$\lim_{k\to\infty}\big(\widehat{Y}^{n,k}_t-Y^{n,k}_t\big)=\lim_{k\to\infty}\mathbf{1}_{\{t\ge\wp_n+2^{1-k}\}}(U_{(\wp_n+2^{1-k})\wedge T}-U_t)=\mathbf{1}_{\{t>\wp_n\}}(U_{\wp_n}-U_t)$$

by the continuity of U, we see that $\mathscr{Y}^n_t:=\lim\limits_{k\to\infty}\widehat{Y}^{n,k}_t=\mathbf{1}_{\{t\le\wp_n\}}L_t+\mathbf{1}_{\{t>\wp_n\}}U_{\wp_n}$, $\forall\,t\in[0,T]$, which is an $\mathbf{F}-$adapted process with all càglàd paths. For any $(t,\omega)\in[0,T]\times\Omega$, Proposition 2.1 (3) shows that $(\mathscr{Y}^n)^{t,\omega}$ is an $\mathbf{F}^t-$adapted process with all càglàd paths and thus an $\mathbf{F}^t-$progressively measurable process. Then we can consider the following $\overline{\mathscr{E}}-$Snell envelope of $\mathscr{Y}^n$:

$$\mathscr{Z}^n_t(\omega):=\sup_{(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\big[(\mathscr{Y}^n)^{t,\omega}_\gamma\big],\qquad\forall\,(t,\omega)\in[0,T]\times\Omega.$$

Again, $\mathscr{Y}^n$ and $\mathscr{Z}^n$ are bounded by M_0.

The next two inequalities show how $Z^{n,k}$ converges to $\mathscr{Z}^n$ in term of 2^{1-k} and how $\mathscr{Z}^n$ differs from $\mathscr{Z}^{n+1}$, both inequalities also depend on the historical path of process U.

Proposition 5.2. *Assume* (3.1), *(A1), (A2), (P2) and let $n,k\in\mathbb{N}$. It holds for any $(t,\omega)\in[0,T]\times\Omega$ that*

$$-2\widehat{\rho}_0\big(2^{1-k}\big)\;\le\;Z^{n,k}_t(\omega)-\mathscr{Z}^n_t(\omega)-U\big((\wp_n(\omega)+2^{1-k})\wedge t,\omega\big)+U(\wp_n(\omega)\wedge t,\omega)\le\widehat{\rho}_0\big(2^{1-k}\big)\tag{5.4}$$

$$\text{and}\quad -2\widehat{\rho}_0\big(\tfrac{2T}{n+3}\big)\;\le\;\mathscr{Z}^{n+1}_t(\omega)-\mathscr{Z}^n_t(\omega)-U\big(\wp_{n+1}(\omega)\wedge t,\omega\big)+U\big(\wp_n(\omega)\wedge t,\omega\big)\le\widehat{\rho}_0\big(\tfrac{2T}{n+3}\big).\tag{5.5}$$

As $\widehat{\rho}_0$ satisfies (2.4) with some $\widehat{\mathfrak{C}}>0$ and $1<\widehat{\mathfrak{p}}_1\le\widehat{\mathfrak{p}}_2$ by (P2), we see from (5.5) that for each $(t,\omega)\in[0,T]\times\Omega$, $\{\mathscr{Z}^n_t(\omega)\}_{n\in\mathbb{N}}$ is a Cauchy sequence, and thus admits a limit $\mathscr{Z}_t(\omega)$. The following results shows that $\mathscr{Z}$ is an $\mathbf{F}-$adapted continuous process above the Snell envelope of the stopped payoff process $\mathscr{Y}^{\tau_0}$ and that the first time $\mathscr{Z}$ meets $\mathscr{Y}$ is exactly the optimal stopping time expected in Theorem 3.1.

Proposition 5.3. *Assume* (3.1), *(A1), (A2) and (P2)−(P4).*
(1) For any $n\in\mathbb{N}$, $\mathscr{Z}^n$ is an $\mathbf{F}-$adapted process bounded by M_0 that has all continuous paths.
(2) For any $(t,\omega)\in[0,T]\times\Omega$, the limit $\mathscr{Z}_t(\omega):=\lim\limits_{n\to\infty}\mathscr{Z}^n_t(\omega)$ exists and satisfies

$$-2\varepsilon_n\le\mathscr{Z}_t(\omega)-\mathscr{Z}^n_t(\omega)-U\big(\tau_0(\omega)\wedge t,\omega\big)+U\big(\wp_n(\omega)\wedge t,\omega\big)\le\varepsilon_n,\qquad\forall\,n\in\mathbb{N},\tag{5.6}$$

where $\varepsilon_n:=\sum_{i=n}^\infty\widehat{\rho}_0\big(\frac{2T}{i+3}\big)$ decreases to 0 as $n\to\infty$.

(*3*) $\mathscr{Z}$ *is an* $\mathbf{F}-$*adapted process bounded by* M_0 *that has all continuous paths. Set* $\widehat{\mathscr{Y}_t}:=\mathscr{Y}_{\tau_0\wedge t}$, $t\in[0,T]$. *It holds for any* $\omega\in\Omega$ *that*

$$\widehat{\mathscr{Y}_t}(\omega)\le \sup_{(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\Big[\widehat{\mathscr{Y}}^{\,t,\omega}_{\gamma}\Big]\le \mathscr{Z}_t(\omega),\ \ \forall\, t\in[0,T]\ \text{and}\ \mathscr{Z}\big(t,\omega\big)=U\big(\tau_0(\omega),\omega\big),\ \ \forall\, t\in[\tau_0(\omega),T]. \tag{5.7}$$

(*4*) $\gamma_*:=\inf\big\{t\in[0,T]:\,\mathscr{Z}_t=\widehat{\mathscr{Y}_t}\big\}=\inf\{t\in[0,\tau_0):\mathscr{Z}_t=L_t\}\wedge\tau_0$ *is an* $\mathbf{F}-$*stopping time.*

6 Proofs

6.1 Proofs of Results in Section 3

Proof of Remark 3.1: 2) Let $(Y,\wp)\in\mathfrak{S}$, $\widetilde{\mathbb{P}}\in\mathfrak{P}_s$ and $\gamma\in\mathcal{T}^s$. Given $\widetilde{\omega}\in\Omega^t$, Lemma A.2 shows that

$$\begin{aligned}\mathbb{E}_{\widetilde{\mathbb{P}}}\Big[\big|\widehat{Y}^{s,\omega\otimes_t\widetilde{\omega}}_{\gamma}-\widehat{Y}^{s,\omega\otimes_t\widetilde{\omega}'}_{\gamma}\big|\Big] &\le \widehat{\rho}_Y\Big((1+\kappa_\wp)\|\omega\otimes_t\widetilde{\omega}-\omega\otimes_t\widetilde{\omega}'\|_{0,s}+\phi^{\omega\otimes_t\widetilde{\omega}}_s\big(\kappa_\wp\|\omega\otimes_t\widetilde{\omega}-\omega\otimes_t\widetilde{\omega}'\|_{0,s}\big)\Big)\\ &\le \widehat{\rho}_Y\Big((1+\kappa_\wp)\|\widetilde{\omega}-\widetilde{\omega}'\|_{t,T}+\phi^{\omega\otimes_t\widetilde{\omega}}_s\big(\kappa_\wp\|\widetilde{\omega}-\widetilde{\omega}'\|_{t,T}\big)\Big),\quad \forall\,\widetilde{\omega}'\in\Omega^t.\end{aligned}$$

Hence, the mapping $\widetilde{\omega}\to\mathbb{E}_{\widetilde{\mathbb{P}}}\Big[\widehat{Y}^{s,\omega\otimes_t\widetilde{\omega}}_{\gamma}\Big]$ is continuous under norm $\|\ \|_{t,T}$ and thus $\mathcal{F}^t_T-$measurable.
3) Similar to the proof of Remark 3.3 (2) in [6], one can show that the probability $\widehat{\mathbb{P}}$ defined in (3.7) satisfies (P4) (i) and the first part of (P4) (ii): i.e. $\widehat{\mathbb{P}}(A\cap\mathcal{A}_0)=\mathbb{P}(A\cap\mathcal{A}_0)$, $\forall\,A\in\mathcal{F}^t_T$, and $\widehat{\mathbb{P}}(A\cap\mathcal{A}_j)=\mathbb{P}(A\cap\mathcal{A}_j)$, $\forall\,j=1,\cdots,\lambda$, $\forall\,A\in\mathcal{F}^t_s$.

To see $\widehat{\mathbb{P}}$ satisfying (3.6) for some $(Y,\wp)\in\mathfrak{S}$, we fix $j=1,\cdots,\lambda$, $A\in\mathcal{F}^t_s$, and $\gamma\in\mathcal{T}^s$. By Lemma 2.1, $(A\cap\mathcal{A}_j)^{s,\widetilde{\omega}}=\Omega^s$ (resp. $=\emptyset$), when $\widetilde{\omega}\in A\cap\mathcal{A}_j$ (resp. $\notin A\cap\mathcal{A}_j$). Then we can deduce that

$$\begin{aligned}\mathbb{E}_{\widehat{\mathbb{P}}}\Big[\mathbf{1}_{A\cap\mathcal{A}_j}\widehat{Y}^{t,\omega}_{\gamma(\Pi^t_s)}\Big] &= \sum_{j'=1}^{\lambda}\mathbb{E}_{\mathbb{P}}\bigg[\mathbf{1}_{\{\widetilde{\omega}\in\mathcal{A}_{j'}\}}\mathbb{E}_{\mathbb{P}_{j'}}\Big[\Big(\mathbf{1}_{A\cap\mathcal{A}_j}\widehat{Y}^{t,\omega}_{\gamma(\Pi^t_s)}\Big)^{s,\widetilde{\omega}}\Big]\bigg]=\sum_{j'=1}^{\lambda}\mathbb{E}_{\mathbb{P}}\bigg[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}_j\}}\mathbf{1}_{\{\widetilde{\omega}\in\mathcal{A}_{j'}\}}\mathbb{E}_{\mathbb{P}_{j'}}\Big[\Big(\widehat{Y}^{t,\omega}_{\gamma(\Pi^t_s)}\Big)^{s,\widetilde{\omega}}\Big]\bigg]\\ &= \mathbb{E}_{\mathbb{P}}\bigg[\mathbf{1}_{\{\widetilde{\omega}\in A\cap\mathcal{A}_j\}}\mathbb{E}_{\mathbb{P}_j}\Big[\widehat{Y}^{s,\omega\otimes_t\widetilde{\omega}}_{\gamma}\Big]\bigg],\end{aligned}$$

where we used the fact that for any $\widehat{\omega}\in\Omega^s$, $\Big(\widehat{Y}^{t,\omega}_{\gamma(\Pi^t_s)}\Big)^{s,\widetilde{\omega}}(\widehat{\omega})=\Big(\widehat{Y}^{t,\omega}_{\gamma(\Pi^t_s)}\Big)(\widetilde{\omega}\otimes_s\widehat{\omega})=\widehat{Y}\big(\gamma\big(\Pi^t_s(\widetilde{\omega}\otimes_s\widehat{\omega})\big),\omega\otimes_t(\widetilde{\omega}\otimes_s\widehat{\omega})\big)=$ $\widehat{Y}\big(\gamma(\widehat{\omega}),(\omega\otimes_t\widetilde{\omega})\otimes_s\widehat{\omega}\big)=\widehat{Y}^{s,\omega\otimes_t\widetilde{\omega}}_{\gamma}(\widehat{\omega})$. □

Proof of Example 3.3: Let $\rho\in\mathfrak{M}$ satisfies (2.4) with some $C>0$ and $0<p_1\le p_2$. Fix $t\in[0,T)$ and $\delta\in(0,\infty)$. We consider an enlarged canonical space $\overline{\Omega}^t:=\Omega^t\times\Omega^t\times\Omega^t$ with canonical processes

$$\overline{B}_t(\overline{\omega})=\big(X_t(\overline{\omega}),A_t(\overline{\omega}),M_t(\overline{\omega})\big)=\big(x(t),a(t),m(t)\big),\quad \forall\,\overline{\omega}=(x,a,m)\in\overline{\Omega}^t,\ \ \forall\,t\in[0,T].$$

Given $\mathbb{P}\in\mathcal{P}^\ell_t$, there exists an extension $\overline{\mathbb{P}}$ of $\mathbb{P}$ on $\overline{\Omega}^t$ such that
(i) $\overline{\mathbb{P}}\big\{\overline{\omega}\in\overline{\Omega}^t:X(\overline{\omega})\in A\big\}=\mathbb{P}(A)$ for any $A\in\mathcal{F}^t_T$;
(ii) $X=K+M$, $\overline{\mathbb{P}}-$a.s., in which K is an absolutely continuous process with $\big|\frac{dK_t}{dt}\big|\le\ell$, $\overline{\mathbb{P}}-$a.s., and M is a $\overline{\mathbb{P}}-$martingale with $\mathrm{trace}\big(\frac{d\langle M\rangle_t}{dt}\big)\le 2\ell$, $\overline{\mathbb{P}}-$a.s.

Let $\zeta\in\mathcal{T}^t$ and set $\eta:=\sup\limits_{r\in[\zeta(X),\,(\zeta(X)+\delta)\wedge T]}\big|M_r-M_{\zeta(X)}\big|=\sup\limits_{r\in[t,T]}\big|M_{(\zeta(X)+\delta)\wedge r}-M_{\zeta(X)\wedge r}\big|$. Given $p>0$, since

$$\big(1\wedge n^{p-1}\big)\sum_{i=1}^{n}a_i^p\le\Big(\sum_{i=1}^{n}a_i\Big)^p\le\big(1\vee n^{p-1}\big)\sum_{i=1}^{n}a_i^p,\quad \forall\,n\in\mathbb{N},\ \ \forall\,\{a_i\}_{i=1}^n\subset[0,\infty), \tag{6.1}$$

one can deduce from the Burkholder-Davis-Gundy inequality that

$$\begin{aligned}\mathbb{E}_{\overline{\mathbb{P}}}\big[\eta^p\big] &\le \big(1\vee d^{\frac{p}{2}-1}\big)\sum_{i=1}^{d}\mathbb{E}_{\overline{\mathbb{P}}}\Big[\sup_{r\in[t,T]}\big|M^i_{(\zeta(X)+\delta)\wedge r}-M^i_{\zeta(X)\wedge r}\big|^p\Big]\le c_p\big(1\vee d^{\frac{p}{2}-1}\big)\sum_{i=1}^{d}\mathbb{E}_{\overline{\mathbb{P}}}\Big[\Big(\int_t^T\mathbf{1}_{\{\zeta(X)\le r\le\zeta(X)+\delta\}}d\langle M^i,M^i\rangle_r\Big)^{\frac{p}{2}}\Big]\\ &\le c_p\frac{1\vee d^{\frac{p}{2}-1}}{1\wedge d^{\frac{p}{2}-1}}\mathbb{E}_{\overline{\mathbb{P}}}\Big[\Big(\int_t^T\mathbf{1}_{\{\zeta(X)\le r\le\zeta(X)+\delta\}}\mathrm{trace}\Big(\frac{d\langle M\rangle_r}{dr}\Big)dr\Big)^{\frac{p}{2}}\Big]\le c_p\frac{1\vee d^{\frac{p}{2}-1}}{1\wedge d^{\frac{p}{2}-1}}(2\ell\delta)^{\frac{p}{2}},\end{aligned}$$

where c_p is a constant depending on p. Then we see from (i), (ii) and (6.1) that

$$\mathbb{E}_{\mathbb{P}}\Big[\rho\Big(\delta+\sup_{r\in[\zeta,(\zeta+\delta)\wedge T]}\big|B^t_r-B^t_\zeta\big|\Big)\Big]=\mathbb{E}_{\overline{\mathbb{P}}}\Big[\rho\Big(\delta+\sup_{r\in[\zeta(X),(\zeta(X)+\delta)\wedge T]}\big|X_r-X_{\zeta(X)}\big|\Big)\Big]\le C\sum_{i=1}^{2}\mathbb{E}_{\overline{\mathbb{P}}}\Big[\big((1+\ell)\delta+\eta\big)^{p_i}\Big]$$
$$\le C\sum_{i=1}^{2}(1\vee 2^{p_i-1})\big((1+\ell)^{p_i}\delta^{p_i}+\mathbb{E}_{\overline{\mathbb{P}}}[\eta^{p_i}]\big)\le C(1\vee 2^{p_2-1})\sum_{i=1}^{2}\Big((1+\ell)^{p_i}\delta^{p_i}+\delta^{p_i/2}+\delta^{-1/2}\mathbb{E}_{\overline{\mathbb{P}}}\big[\mathbf{1}_{\{\eta\ge\sqrt{\delta}\}}\eta^{1+p_i}\big]\Big)$$
$$\le\frac14\widehat{C}\sum_{i=1}^{2}(\delta^{p_i}+\delta^{p_i/2})\le\widehat{C}\big(\delta^{p_1/2}\vee\delta^{p_2}\big)$$

for some constant $\widehat{C}$ depending on C, d, ℓ, p_1 , p_2 and c_{p_2}. Hence, (3.5) holds for $\widehat{\rho}(\delta):=\widehat{C}\big(\delta^{p_1/2}\vee\delta^{p_2}\big)$. □

6.2 Proofs of Results in Section 4

Proof of Proposition 4.1: Fix $(Y,\wp)\in\mathfrak{S}$ and $(t,\omega)\in[0,T]\times\Omega$. Let $\omega'\in\Omega$. We set $t_1:=\wp(\omega)\wedge\wp(\omega')\wedge t$, $t_2:=(\wp(\omega)\vee\wp(\omega'))\wedge t$. Given $(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t$, we see from Lemma A.2 that

$$\mathbb{E}_{\mathbb{P}}\big[\widehat{Y}^{t,\omega}_\gamma\big]-Z_t(\omega')\le\mathbb{E}_{\mathbb{P}}\big[\widehat{Y}^{t,\omega}_\gamma-\widehat{Y}^{t,\omega'}_\gamma\big]\le\widehat{\rho}_Y\Big((1+\kappa_\wp)\|\omega-\omega'\|_{0,t}+\sup_{r\in[t_1,t_2]}\big|\omega(r)-\omega(t_1)\big|\Big),$$
$$\text{and}\quad\mathbb{E}_{\mathbb{P}}\big[\widehat{Y}^{t,\omega'}_\gamma\big]-Z_t(\omega)\le\mathbb{E}_{\mathbb{P}}\big[\widehat{Y}^{t,\omega'}_\gamma-\widehat{Y}^{t,\omega}_\gamma\big]\le\widehat{\rho}_Y\Big((1+\kappa_\wp)\|\omega-\omega'\|_{0,t}+\sup_{r\in[t_1,t_2]}\big|\omega(r)-\omega(t_1)\big|\Big).$$

Taking supremum over $(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t$ on the left-hand-sides of both inequalities leads to (4.2).

For any $\varepsilon>0$, there exists a $\lambda>0$ such that $\widehat{\rho}_Y(x)<\varepsilon,\ \forall\,x\in[0,\lambda)$. One can also find a $\widetilde{\lambda}(t,\omega)>0$ such that $\phi^\omega_t(y)<\lambda/2,\ \forall\,y\in\big[0,\widetilde{\lambda}(t,\omega)\big)$. Now, taking $\delta(t,\omega):=\frac{\lambda}{2(1+\kappa_\wp)}\wedge\frac{\widetilde{\lambda}(t,\omega)}{\kappa_\wp}$, we will obtain (4.3). □

Proof of Proposition 4.2: Fix $(Y,\wp)\in\mathfrak{S}$.

1) *We first show* (4.4) *for stopping time ν taking finitely many values.*

Fix $(t,\omega)\in[0,T]\times\Omega$ and let $\nu\in\mathcal{T}^t$ take values in some finite subset $\{t_1<\cdots<t_m\}$ of $[t,T]$. We simply denote

$$\mathcal{Y}_r:=\widehat{Y}^{t,\omega}_r\quad\text{and}\quad\mathcal{Z}_r:=Z^{t,\omega}_r,\qquad\forall\,r\in[t,T].\tag{6.2}$$

Proposition 2.1 (3) and (3.4) show that $\mathcal{Y}$ is an $\mathbf{F}^t-$adapted bounded process with all continuous paths.

1a) *In the first step, we show*

$$Z_t(\omega)\le\sup_{(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\gamma<\nu\}}\mathcal{Y}_\gamma+\mathbf{1}_{\{\gamma\ge\nu\}}\mathcal{Z}_\nu\big]\tag{6.3}$$

for the $\mathbf{F}^t-$stopping time ν taking finitely many values.

Let $(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t$ and let $i=1,\cdots,m$. In light of (2.7), there exists a $\mathbb{P}-$null set $\mathcal{N}_i$ such that

$$\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\gamma\vee t_i}\big|\mathcal{F}^t_{t_i}\big](\widetilde{\omega})=\mathbb{E}_{\mathbb{P}^{t_i,\widetilde{\omega}}}\Big[\big(\mathcal{Y}_{\gamma\vee t_i}\big)^{t_i,\widetilde{\omega}}\Big]=\mathbb{E}_{\mathbb{P}^{t_i,\widetilde{\omega}}}\Big[\widehat{Y}^{t_i,\omega\otimes_t\widetilde{\omega}}_{(\gamma\vee t_i)^{t_i,\widetilde{\omega}}}\Big],\qquad\forall\,\widetilde{\omega}\in\mathcal{N}^c_i,\tag{6.4}$$

where we used the fact that for any $\widetilde{\omega}\in\Omega^t$ and $\widehat{\omega}\in\Omega^{t_i}$

$$\big(\mathcal{Y}_{\gamma\vee t_i}\big)^{t_i,\widetilde{\omega}}(\widehat{\omega})=\mathcal{Y}_{\gamma\vee t_i}(\widetilde{\omega}\otimes_{t_i}\widehat{\omega})=\widehat{Y}\big((\gamma\vee t_i)(\widetilde{\omega}\otimes_{t_i}\widehat{\omega}),\omega\otimes_t(\widetilde{\omega}\otimes_{t_i}\widehat{\omega})\big)=\widehat{Y}\big((\gamma\vee t_i)^{t_i,\widetilde{\omega}}(\widehat{\omega}),(\omega\otimes_t\widetilde{\omega})\otimes_{t_i}\widehat{\omega}\big)=\widehat{Y}^{t_i,\omega\otimes_t\widetilde{\omega}}_{(\gamma\vee t_i)^{t_i,\widetilde{\omega}}}(\widehat{\omega}).$$

By (P3), there exist an extension $(\Omega^t,\mathcal{F}^{(i)},\mathbb{P}^{(i)})$ of $(\Omega^t,\mathcal{F}^t_T,\mathbb{P})$ and $\Omega^{(i)}\in\mathcal{F}^{(i)}$ with $\mathbb{P}^{(i)}(\Omega^{(i)})=1$ such that for any $\widetilde{\omega}\in\Omega^{(i)}$, $\mathbb{P}^{t_i,\widetilde{\omega}}\in\mathcal{P}_{t_i}$. Given $\widetilde{\omega}\in\Omega^{(i)}\cap\mathcal{N}^c_i$, since $(\gamma\vee t_i)^{t_i,\widetilde{\omega}}\in\mathcal{T}^{t_i}$ by Proposition 2.1 (2), we see from (6.4) that

$$\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\gamma\vee t_i}\big|\mathcal{F}^t_{t_i}\big](\widetilde{\omega})=\mathbb{E}_{\mathbb{P}^{t_i,\widetilde{\omega}}}\Big[\widehat{Y}^{t_i,\omega\otimes_t\widetilde{\omega}}_{(\gamma\vee t_i)^{t_i,\widetilde{\omega}}}\Big]\le Z(t_i,\omega\otimes_t\widetilde{\omega})=\mathcal{Z}_{t_i}(\widetilde{\omega}).$$

So $\Omega^{(i)}\cap\mathcal{N}^c_i\subset A_i:=\{\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\gamma\vee t_i}\big|\mathcal{F}^t_{t_i}\big]\le\mathcal{Z}_{t_i}\}$. The $\mathbf{F}-$adaptedness of Z by Proposition 4.1 as well as Proposition 2.1 (3) imply that $\mathcal{Z}_{t_i}$ is $\mathcal{F}^t_{t_i}-$measurable and thus $A_i\in\mathcal{F}^t_{t_i}$. It follows that $\mathbb{P}(A_i)=\mathbb{P}^{(i)}(A_i)\ge\mathbb{P}^{(i)}\big(\Omega^{(i)}\cap\mathcal{N}^c_i\big)=1$. Namely,

$$\mathbb{E}_{\mathbb{P}}\big[\mathcal{Y}_{\gamma\vee t_i}\big|\mathcal{F}^t_{t_i}\big]\le\mathcal{Z}_{t_i},\quad\mathbb{P}-\text{a.s.}\tag{6.5}$$

Setting $A_i := \{\nu = t_i\} \in \mathcal{F}^t_{t_i}$, as

$$\mathbf{1}_{\{\gamma<t_i\}}\mathcal{Y}_\gamma = \mathbf{1}_{\{\gamma<t_i\}}\mathcal{Y}_{\gamma\wedge t_i} \in \mathcal{F}^t_{\gamma\wedge t_i} \subset \mathcal{F}^t_{t_i}, \tag{6.6}$$

we can deduce from (6.5) that

$$\begin{aligned}\mathbb{E}_\mathbb{P}[\mathbf{1}_{A_i}\mathcal{Y}_\gamma] &= \mathbb{E}_\mathbb{P}\big[\mathbb{E}_\mathbb{P}[\mathbf{1}_{A_i}\mathbf{1}_{\{\gamma<t_i\}}\mathcal{Y}_\gamma+\mathbf{1}_{A_i}\mathbf{1}_{\{\gamma\ge t_i\}}\mathcal{Y}_{\gamma\vee t_i}|\mathcal{F}^t_{t_i}]\big]=\mathbb{E}_\mathbb{P}\big[\mathbf{1}_{A_i}\mathbf{1}_{\{\gamma<t_i\}}\mathcal{Y}_\gamma+\mathbf{1}_{A_i}\mathbf{1}_{\{\gamma\ge t_i\}}\mathbb{E}_\mathbb{P}[\mathcal{Y}_{\gamma\vee t_i}|\mathcal{F}^t_{t_i}]\big]\\ &\le \mathbb{E}_\mathbb{P}\big[\mathbf{1}_{A_i}\mathbf{1}_{\{\gamma<t_i\}}\mathcal{Y}_\gamma+\mathbf{1}_{A_i}\mathbf{1}_{\{\gamma\ge t_i\}}\mathcal{Z}_{t_i}\big] = \mathbb{E}_\mathbb{P}\big[\mathbf{1}_{A_i}\mathbf{1}_{\{\gamma<\nu\}}\mathcal{Y}_\gamma+\mathbf{1}_{A_i}\mathbf{1}_{\{\gamma\ge\nu\}}\mathcal{Z}_\nu\big],\end{aligned}$$

and similarly that $\mathbb{E}_\mathbb{P}[\mathbf{1}_{A_i}\mathcal{Y}_\gamma]=\mathbb{E}_\mathbb{P}\big[\mathbf{1}_{A_i}\mathbf{1}_{\{\gamma\le t_i\}}\mathcal{Y}_\gamma+\mathbf{1}_{A_i}\mathbf{1}_{\{\gamma>t_i\}}\mathbb{E}_\mathbb{P}[\mathcal{Y}_{\gamma\vee t_i}|\mathcal{F}^t_{t_i}]\big]\le\mathbb{E}_\mathbb{P}\big[\mathbf{1}_{A_i}\mathbf{1}_{\{\gamma\le\nu\}}\mathcal{Y}_\gamma+\mathbf{1}_{A_i}\mathbf{1}_{\{\gamma>\nu\}}\mathcal{Z}_\nu\big]$. Summing them up over $i\in\{1,\cdots,m\}$ yields that

$$\mathbb{E}_\mathbb{P}[\mathcal{Y}_\gamma]\le\mathbb{E}_\mathbb{P}\big[\mathbf{1}_{\{\gamma<\nu\}}\mathcal{Y}_\gamma+\mathbf{1}_{\{\gamma\ge\nu\}}\mathcal{Z}_\nu\big] \quad \text{and} \quad \mathbb{E}_\mathbb{P}[\mathcal{Y}_\gamma]\le\mathbb{E}_\mathbb{P}\big[\mathbf{1}_{\{\gamma\le\nu\}}\mathcal{Y}_\gamma+\mathbf{1}_{\{\gamma>\nu\}}\mathcal{Z}_\nu\big]. \tag{6.7}$$

Taking supremum of the former over $(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t$ leads to (6.3).
1b) *To demonstrate the inverse inequality of* (6.3), *we shall paste the local approximating* $\mathbb{P}-$*maximizers of* $Z^{t,\omega}_{t_i}$*'s according to (P4) and then make some estimations.*

Fix $(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t$, $\varepsilon>0$ and let $\delta\in\mathbb{Q}_+$ satisfy $\overline{\rho}_Y(\delta)<\varepsilon/4$. For any $\widetilde{\omega}\in\Omega^t$, let $\delta(\widetilde{\omega})\in\big((0,\delta]\cap\mathbb{Q}\big)\cup\{\delta\}$ such that

$$\widehat{\rho}_Y\Big((1+\kappa_\wp)\delta(\widetilde{\omega})+\phi^{\omega\otimes_t\widetilde{\omega}}_T\big(\kappa_\wp\,\delta(\widetilde{\omega})\big)\Big) < \varepsilon/4. \tag{6.8}$$

Since the canonical space Ω^t is separable and thus Lindelöf, there exists a sequence $\{\widetilde{\omega}_j\}_{j\in\mathbb{N}}$ of Ω^t such that $\underset{j\in\mathbb{N}}{\cup} O_{\delta_j}(\widetilde{\omega}_j)=\Omega^t$ with $\delta_j := \delta(\widetilde{\omega}_j)$.

Let $i=1,\cdots,m$ and $j\in\mathbb{N}$. By (2.1), $\mathcal{A}^i_j := \{\nu=t_i\}\cap\Big(O^{t_i}_{\delta_j}(\widetilde{\omega}_j)\backslash \underset{j'<j}{\cup} O^{t_i}_{\delta_{j'}}(\widetilde{\omega}_{j'})\Big)\in\mathcal{F}^t_{t_i}$. We can find a pair $(\mathbb{P}^i_j,\gamma^i_j)\in\mathcal{P}_{t_i}\times\mathcal{T}^{t_i}$ such that

$$Z_{t_i}(\omega\otimes_t\widetilde{\omega}_j) \le \mathbb{E}_{\mathbb{P}^i_j}\Big[\widehat{Y}^{t_i,\omega\otimes_t\widetilde{\omega}_j}_{\gamma^i_j}\Big] + \varepsilon/4. \tag{6.9}$$

Given $\widetilde{\omega}\in O^{t_i}_{\delta_j}(\widetilde{\omega}_j)$, applying Lemma A.2 with $(t,\omega,\omega',\mathbb{P},\gamma)=\big(t_i,\omega\otimes_t\widetilde{\omega}_j,\omega\otimes_t\widetilde{\omega},\mathbb{P}^i_j,\gamma^i_j\big)$, we see from (6.8) that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}^i_j}\Big[\widehat{Y}^{t_i,\omega\otimes_t\widetilde{\omega}_j}_{\gamma^i_j}-\widehat{Y}^{t_i,\omega\otimes_t\widetilde{\omega}}_{\gamma^i_j}\Big] &\le \widehat{\rho}_Y\Big((1+\kappa_\wp)\|\omega\otimes_t\widetilde{\omega}_j-\omega\otimes_t\widetilde{\omega}\|_{0,t_i}+\phi^{\omega\otimes_t\widetilde{\omega}_j}_{t_i}\big(\kappa_\wp\|\omega\otimes_t\widetilde{\omega}_j-\omega\otimes_t\widetilde{\omega}\|_{0,t_i}\big)\Big)\\ &=\widehat{\rho}_Y\Big((1+\kappa_\wp)\|\widetilde{\omega}_j-\widetilde{\omega}\|_{t,t_i}+\phi^{\omega\otimes_t\widetilde{\omega}_j}_{t_i}\big(\kappa_\wp\|\widetilde{\omega}_j-\widetilde{\omega}\|_{t,t_i}\big)\Big)\le\widehat{\rho}_Y\Big((1+\kappa_\wp)\delta_j+\phi^{\omega\otimes_t\widetilde{\omega}_j}_T\big(\kappa_\wp\,\delta_j\big)\Big)<\varepsilon/4.\end{aligned}$$

Then applying (4.2) with $(t,\omega,\omega')=\big(t_i,\omega\otimes_t\widetilde{\omega}_j,\omega\otimes_t\widetilde{\omega}\big)$, one can deduce from (6.9) and (6.8) again that

$$\begin{aligned}\mathcal{Z}_{t_i}(\widetilde{\omega}) &= Z_{t_i}(\omega\otimes_t\widetilde{\omega}) \le Z_{t_i}\big(\omega\otimes_t\widetilde{\omega}_j\big)+\widehat{\rho}_Y\Big((1+\kappa_\wp)\|\omega\otimes_t\widetilde{\omega}_j-\omega\otimes_t\widetilde{\omega}\|_{0,t_i}+\phi^{\omega\otimes_t\widetilde{\omega}_j}_{t_i}\big(\kappa_\wp\|\omega\otimes_t\widetilde{\omega}_j-\omega\otimes_t\widetilde{\omega}\|_{0,t_i}\big)\Big)\\ &= Z_{t_i}(\omega\otimes_t\widetilde{\omega}_j)+\widehat{\rho}_Y\Big((1+\kappa_\wp)\|\widetilde{\omega}_j-\widetilde{\omega}\|_{t,t_i}+\phi^{\omega\otimes_t\widetilde{\omega}_j}_{t_i}\big(\kappa_\wp\|\widetilde{\omega}_j-\widetilde{\omega}\|_{t,t_i}\big)\Big)\le Z_{t_i}(\omega\otimes_t\widetilde{\omega}_j)+\widehat{\rho}_Y\Big((1+\kappa_\wp)\delta_j+\phi^{\omega\otimes_t\widetilde{\omega}_j}_T\big(\kappa_\wp\,\delta_j\big)\Big)\\ &< \mathbb{E}_{\mathbb{P}^i_j}\Big[\widehat{Y}^{t_i,\omega\otimes_t\widetilde{\omega}_j}_{\gamma^i_j}\Big]+\varepsilon/2<\mathbb{E}_{\mathbb{P}^i_j}\Big[\widehat{Y}^{t_i,\omega\otimes_t\widetilde{\omega}}_{\gamma^i_j}\Big]+\frac{3}{4}\varepsilon.\end{aligned} \tag{6.10}$$

Now, fix $\lambda\in\mathbb{N}$. Setting $\mathbb{P}^\lambda_{m+1}:=\mathbb{P}$, we recursively pick up $\mathbb{P}^\lambda_i$, $i=m,\cdots,1$ from $\mathcal{P}_t$ such that (P4) holds for $\Big(s,\widehat{\mathbb{P}},\mathbb{P},\big\{(\mathcal{A}_j,\delta_j,\widetilde{\omega}_j,\mathbb{P}_j)\big\}_{j=1}^\lambda\Big)=\Big(t_i,\mathbb{P}^\lambda_i,\mathbb{P}^\lambda_{i+1},\big\{(\mathcal{A}^i_j,\delta_j,\widetilde{\omega}_j,\mathbb{P}^i_j)\big\}_{j=1}^\lambda\Big)$ and $\mathcal{A}_0=\mathcal{A}^i_0:=\Big(\overset{\lambda}{\underset{j=1}{\cup}}\mathcal{A}^i_j\Big)^c\in\mathcal{F}^t_{t_i}$. Then

$$\mathbb{E}_{\mathbb{P}^\lambda_i}[\xi]=\mathbb{E}_{\mathbb{P}^\lambda_{i+1}}[\xi],\ \ \forall\,\xi\in L^1(\mathcal{F}^t_{t_i},\mathbb{P}^\lambda_i)\cap L^1\big(\mathcal{F}^t_{t_i},\mathbb{P}^\lambda_{i+1}\big) \text{ and } \mathbb{E}_{\mathbb{P}^\lambda_i}[\mathbf{1}_{\mathcal{A}^i_0}\xi]=\mathbb{E}_{\mathbb{P}^\lambda_{i+1}}[\mathbf{1}_{\mathcal{A}^i_0}\xi],\ \ \forall\,\xi\in L^1(\mathcal{F}^t_T,\mathbb{P}^\lambda_i)\cap L^1\big(\mathcal{F}^t_T,\mathbb{P}^\lambda_{i+1}\big). \tag{6.11}$$

For any $i=1,\cdots,m$, as Lemma A.1 of [6] shows that $\gamma^i_j(\Pi^t_{t_i})\in\mathcal{T}^t_{t_i}$, stitching γ with $\gamma^i_j(\Pi^t_{t_i})$'s forms a new $\mathbf{F}^t-$stopping time

$$\widehat{\gamma}_\lambda:=\mathbf{1}_{\{\gamma<\nu\}}\gamma+\mathbf{1}_{\{\gamma\ge\nu\}}\Big(\mathbf{1}_{\overset{m}{\underset{i=1}{\cap}}\mathcal{A}^i_0}\gamma+\sum_{i=1}^m\sum_{j=1}^\lambda\mathbf{1}_{\mathcal{A}^i_j}\gamma^i_j(\Pi^t_{t_i})\Big). \tag{6.12*}$$

We see from (6.11) that

$$\mathbb{E}_{\mathbb{P}^\lambda_1}\Big[\mathbf{1}_{\underset{i=1}{\overset{m}{\cap}}\mathcal{A}^i_0}\mathcal{Y}_{\widehat{\gamma}_\lambda}\Big]=\mathbb{E}_{\mathbb{P}^\lambda_2}\Big[\mathbf{1}_{\underset{i=1}{\overset{m}{\cap}}\mathcal{A}^i_0}\mathcal{Y}_{\widehat{\gamma}_\lambda}\Big]=\cdots=\mathbb{E}_{\mathbb{P}^\lambda_m}\Big[\mathbf{1}_{\underset{i=1}{\overset{m}{\cap}}\mathcal{A}^i_0}\mathcal{Y}_{\widehat{\gamma}_\lambda}\Big]=\mathbb{E}_{\mathbb{P}^\lambda_{m+1}}\Big[\mathbf{1}_{\underset{i=1}{\overset{m}{\cap}}\mathcal{A}^i_0}\mathcal{Y}_{\widehat{\gamma}_\lambda}\Big]=\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\underset{i=1}{\overset{m}{\cap}}\mathcal{A}^i_0}\mathcal{Y}_{\gamma}\Big]. \tag{6.13}$$

On the other hand, for any $(i,j)\in\{1,\cdots,m\}\times\{1,\cdots,\lambda\}$, as $\mathcal{A}^i_j\subset\mathcal{A}^{i'}_0$ for $i'\in\{1,\cdots,m\}\backslash\{i\}$, we can deduce from (6.6), (6.11), (3.6) and (6.10) that

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}^\lambda_1}\big[\mathbf{1}_{\mathcal{A}^i_j}\mathcal{Y}_{\widehat{\gamma}_\lambda}\big]&=\mathbb{E}_{\mathbb{P}^\lambda_2}\Big[\mathbf{1}_{\mathcal{A}^i_j}\mathcal{Y}_{\widehat{\gamma}_\lambda}\Big]=\cdots=\mathbb{E}_{\mathbb{P}^\lambda_{i-1}}\big[\mathbf{1}_{\mathcal{A}^i_j}\mathcal{Y}_{\widehat{\gamma}_\lambda}\big]=\mathbb{E}_{\mathbb{P}^\lambda_i}\big[\mathbf{1}_{\mathcal{A}^i_j}\mathcal{Y}_{\widehat{\gamma}_\lambda}\big]=\mathbb{E}_{\mathbb{P}^\lambda_i}\Big[\mathbf{1}_{\{\gamma<\nu\}\cap\mathcal{A}^i_j}\mathcal{Y}_\gamma+\mathbf{1}_{\{\gamma\ge\nu\}\cap\mathcal{A}^i_j}\mathcal{Y}_{\gamma^i_j(\Pi^t_{t_i})}\Big]\\
&=\mathbb{E}_{\mathbb{P}^\lambda_i}\Big[\mathbf{1}_{\{\gamma<t_i\}\cap\mathcal{A}^i_j}\mathcal{Y}_\gamma+\mathbf{1}_{\{\gamma\ge t_i\}\cap\mathcal{A}^i_j}\mathcal{Y}_{\gamma^i_j(\Pi^t_{t_i})}\Big]\ge\mathbb{E}_{\mathbb{P}^\lambda_{i+1}}\Big[\mathbf{1}_{\{\gamma<t_i\}\cap\mathcal{A}^i_j}\mathcal{Y}_\gamma+\mathbf{1}_{\{\gamma(\widetilde{\omega})\ge t_i\}\cap\{\widetilde{\omega}\in\mathcal{A}^i_j\}}\Big(\mathbb{E}_{\mathbb{P}^i_j}\Big[\widehat{Y}^{t_i,\omega\otimes_t\widetilde{\omega}}_{\gamma^i_j}\Big]-\overline{\rho}_Y(\delta)\Big)\Big]\\
&\ge\mathbb{E}_{\mathbb{P}^\lambda_{i+1}}\big[\mathbf{1}_{\{\gamma<t_i\}\cap\mathcal{A}^i_j}\mathcal{Y}_\gamma+\mathbf{1}_{\{\gamma\ge t_i\}\cap\mathcal{A}^i_j}(\mathcal{Z}_{t_i}-\varepsilon)\big]=\cdots=\mathbb{E}_{\mathbb{P}^\lambda_{m+1}}\big[\mathbf{1}_{\{\gamma<\nu\}\cap\mathcal{A}^i_j}\mathcal{Y}_\gamma+\mathbf{1}_{\{\gamma\ge\nu\}\cap\mathcal{A}^i_j}\big(\mathcal{Z}_\nu-\varepsilon\big)\big]\\
&=\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\gamma<\nu\}\cap\mathcal{A}^i_j}\mathcal{Y}_\gamma+\mathbf{1}_{\{\gamma\ge\nu\}\cap\mathcal{A}^i_j}\big(\mathcal{Z}_\nu-\varepsilon\big)\big].
\end{aligned}$$

Taking summation over $(i,j)\in\{1,\cdots,m\}\times\{1,\cdots,\lambda\}$ and then combining with (6.13) yield that

$$\begin{aligned}
Z_t(\omega)\;&\ge\;\mathbb{E}_{\mathbb{P}^\lambda_1}\big[\mathcal{Y}_{\widehat{\gamma}_\lambda}\big]\ge\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\gamma<\nu\}}\mathcal{Y}_\gamma+\mathbf{1}_{\{\gamma\ge\nu\}}\mathcal{Z}_\nu\big]+\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\gamma\ge\nu\}}\mathbf{1}_{\underset{i=1}{\overset{m}{\cap}}\mathcal{A}^i_0}(\mathcal{Y}_\gamma-\mathcal{Z}_\nu)\Big]-\varepsilon\\
&\ge\;\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\gamma<\nu\}}\mathcal{Y}_\gamma+\mathbf{1}_{\{\gamma\ge\nu\}}\mathcal{Z}_\nu\Big]-2M_Y\mathbb{P}\Big(\overset{m}{\underset{i=1}{\cap}}\mathcal{A}^i_0\Big)-\varepsilon,
\end{aligned}\tag{6.14}$$

where $\underset{i=1}{\overset{m}{\cap}}\mathcal{A}^i_0=\Big(\underset{i=1}{\overset{m}{\cup}}\underset{j=1}{\overset{\lambda}{\cup}}\mathcal{A}^i_j\Big)^c$. Since $\underset{j\in\mathbb{N}}{\cup}O^{t_i}_{\delta_j}(\widetilde{\omega}_j)\supset\underset{j\in\mathbb{N}}{\cup}O_{\delta_j}(\widetilde{\omega}_j)=\Omega^t$ for each $i\in\{1,\cdots,m\}$, we see that $\underset{i=1}{\overset{m}{\cup}}\underset{j\in\mathbb{N}}{\cup}\mathcal{A}^i_j=\underset{i=1}{\overset{m}{\cup}}\Big[\{\nu=t_i\}\cap\Big(\underset{j\in\mathbb{N}}{\cup}O^{t_i}_{\delta_j}(\widetilde{\omega}_j)\Big)\Big]=\underset{i=1}{\overset{m}{\cup}}\{\nu=t_i\}=\Omega^t$, letting $\lambda\to\infty$ and then letting $\varepsilon\to 0$ in (6.14) yield that

$$Z_t(\omega)\ge\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\gamma<\nu\}}\mathcal{Y}_\gamma+\mathbf{1}_{\{\gamma\ge\nu\}}\mathcal{Z}_\nu\Big]. \tag{6.15}$$

Taking supremum over $(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t$ and combining with (6.3) prove (4.4) for stopping times ν taking finitely many values.

2) *Next, let us show* (4.5) *and thus the continuity of process* Z*.*

Fix $\omega\in\Omega$ and $0\le t\le s\le T$. If $t=s$, then (4.5) trivially holds. So we assume $t<s$.

2a) *Let us start by proving an auxiliary inequality:*

$$\mathbb{E}_{\mathbb{P}}\Big[\big|Z^{t,\omega}_s-Z_s(\omega)\big|\Big]\le 2C_\varrho M_Y\Big((s-t)^{\frac{q_1}{2}}\vee(s-t)^{q_2-\frac{q_1}{2}}\Big)+\widehat{\rho}_Y\big(\delta_{t,s}(\omega)\big)\vee\widehat{\widehat{\rho}}_Y\big(\delta_{t,s}(\omega)\big):=\widehat{\phi}_{t,s}(\omega). \tag{6.16}$$

For any $\widetilde{\omega}\in\Omega^t$, applying (4.2) with $(t,\omega,\omega')=(s,\omega\otimes_t\widetilde{\omega},\omega)$ yields that

$$\big|Z(s,\omega\otimes_t\widetilde{\omega})-Z_s(\omega)\big|\le\widehat{\rho}_Y\Big((1+\kappa_\wp)\|\omega\otimes_t\widetilde{\omega}-\omega\|_{0,s}+\sup_{r\in[s_1(\widetilde{\omega}),s_2(\widetilde{\omega})]}\big|(\omega\otimes_t\widetilde{\omega})(r)-(\omega\otimes_t\widetilde{\omega})(s_1(\widetilde{\omega}))\big|\Big), \tag{6.17}$$

where $s_1(\widetilde{\omega}):=\wp(\omega\otimes_t\widetilde{\omega})\wedge\wp(\omega)\wedge s$ and $s_2(\widetilde{\omega}):=\big(\wp(\omega\otimes_t\widetilde{\omega})\vee\wp(\omega)\big)\wedge s$.

Let $\mathbb{P}\in\mathcal{P}_t$ and set $A:=\Big\{\sup_{r\in[t,s]}|B^t_r|\le(s-t)^{\frac{q_1}{2}}\Big\}$. As $B^t_t=0$, one can deduce from (4.1) and (3.5) that

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{A^c}|Z^{t,\omega}_s-Z_s(\omega)|\big]&\le 2M_Y\mathbb{P}(A^c)\le 2M_Y(s-t)^{-\frac{q_1}{2}}\mathbb{E}_{\mathbb{P}}\Big[\sup_{r\in[t,s]}|B^t_r-B^t_t|\Big]\le 2M_Y(s-t)^{-\frac{q_1}{2}}\mathbb{E}_{\mathbb{P}}\Big[\varrho\Big((s-t)+\sup_{r\in[t,s]}|B^t_r-B^t_t|\Big)\Big]\\
&\le 2M_Y(s-t)^{-\frac{q_1}{2}}\widehat{\varrho}(s-t)\le 2C_\varrho M_Y\Big((s-t)^{\frac{q_1}{2}}\vee(s-t)^{q_2-\frac{q_1}{2}}\Big).
\end{aligned}\tag{6.18}$$

As to $\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_A|Z^{t,\omega}_s-Z_s(\omega)|\big]$, we shall estimate it by two cases on values of $\wp(\omega)$:

(i) When $\wp(\omega)\le t$, let $\widetilde{\omega}\in A$. Applying Lemma A.1 with $(t,s,\tau)=(0,t,\wp)$ yields that $\wp(\omega\otimes_t\widetilde{\omega})=\wp(\omega)$, thus $s_1(\widetilde{\omega})=s_2(\widetilde{\omega})=\wp(\omega)\wedge s=\wp(\omega)$. Since

$$\|\omega\otimes_t\widetilde{\omega}-\omega\|_{0,s}=\sup_{r\in[t,s]}\big|\widetilde{\omega}(r)+\omega(t)-\omega(r)\big|\le\sup_{r\in[t,s]}\big|\widetilde{\omega}(r)\big|+\sup_{r\in[t,s]}\big|\omega(r)-\omega(t)\big|\le(s-t)^{\frac{q_1}{2}}+\sup_{r\in[t,s]}\big|\omega(r)-\omega(t)\big|, \tag{6.19}$$

we can deduce from (6.17) that

$$\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_A\big|Z^{t,\omega}_s-Z_s(\omega)\big|\Big] \le \widehat{\rho}_Y\big(\delta_{t,s}(\omega)\big). \tag{6.20}$$

(ii) When $\wp(\omega)>t$, applying Lemma A.1 again shows that $\wp(\omega\otimes_t\Omega^t)\subset(t,T]$ and that $\zeta:=\wp^{t,\omega}\wedge\wp(\omega)\wedge s\in\mathcal{T}^t$. Let $\widetilde{\omega}\in A$. Since $\zeta(\widetilde{\omega})=\wp(\omega\otimes_t\widetilde{\omega})\wedge\wp(\omega)\wedge s=s_1(\widetilde{\omega})>t$, we have

$$\sup_{r\in[s_1(\widetilde{\omega}),s_2(\widetilde{\omega})]}\big|(\omega\otimes_t\widetilde{\omega})(r)-(\omega\otimes_t\widetilde{\omega})(s_1(\widetilde{\omega}))\big| = \sup_{r\in[s_1(\widetilde{\omega}),s_2(\widetilde{\omega})]}\big|\widetilde{\omega}(r)-\widetilde{\omega}(s_1(\widetilde{\omega}))\big| = \sup_{r\in[\zeta(\widetilde{\omega}),s_2(\widetilde{\omega})]}\big|B^t_r(\widetilde{\omega})-B^t_\zeta(\widetilde{\omega})\big|.$$

By (2.5) and (6.19), $s_2(\widetilde{\omega})-\zeta(\widetilde{\omega})=s_2(\widetilde{\omega})-s_1(\widetilde{\omega})\le\wp(\omega\otimes_t\widetilde{\omega})\vee\wp(\omega)-\wp(\omega\otimes_t\widetilde{\omega})\wedge\wp(\omega)=|\wp(\omega\otimes_t\widetilde{\omega})-\wp(\omega)|\le\kappa_\wp\|\omega\otimes_t\widetilde{\omega}-\omega\|_{0,s}<\delta_{t,s}(\omega)$. So $\sup_{r\in[s_1(\widetilde{\omega}),s_2(\widetilde{\omega})]}\big|(\omega\otimes_t\widetilde{\omega})(r)-(\omega\otimes_t\widetilde{\omega})(s_1(\widetilde{\omega}))\big|\le\sup_{r\in[\zeta(\widetilde{\omega}),(\zeta(\widetilde{\omega})+\delta_{t,s}(\omega))\wedge T]}\big|B^t_r(\widetilde{\omega})-B^t_\zeta(\widetilde{\omega})\big|$. Then (6.17), (6.19) and (3.5) imply that $\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_A\big|Z^{t,\omega}_s-Z_s(\omega)\big|\Big]\le\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_A\widehat{\rho}_Y\Big(\delta_{t,s}(\omega)+\sup_{r\in[\zeta,(\zeta+\delta_{t,s}(\omega))\wedge T]}\big|B^t_r-B^t_\zeta\big|\Big)\Big]\le\widehat{\widehat{\rho}}_Y\big(\delta_{t,s}(\omega)\big)$, which together with (6.18) and (6.20) leads to (6.16).

2b) *Now, we shall use* (6.15), (6.16), (6.7) *as well as* (3.5) *to derive* (4.5).

For any $\mathbb{P}\in\mathcal{P}_t$, applying (6.15) with $\nu=s$ and $\gamma=s$, we see from (6.16) that

$$Z_t(\omega)-Z_s(\omega)\ge\mathbb{E}_{\mathbb{P}}\big[Z^{t,\omega}_s-Z_s(\omega)\big]\ge-\widehat{\phi}_{t,s}(\omega). \tag{6.21}$$

As to the inverse inequality, let us fix $\varepsilon>0$. There exists a pair $(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t$ such that $Z_t(\omega)\le\mathbb{E}_{\mathbb{P}}\big[\widehat{Y}^{t,\omega}_\gamma\big]+\varepsilon$. Applying the first inequality of (6.7) with $\nu=s$ yields that

$$Z_t(\omega)\le\mathbb{E}_{\mathbb{P}}\big[\widehat{Y}^{t,\omega}_\gamma\big]+\varepsilon\le\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\gamma<s\}}\widehat{Y}^{t,\omega}_\gamma+\mathbf{1}_{\{\gamma\ge s\}}Z^{t,\omega}_s\big]+\varepsilon. \tag{6.22}$$

For any $\widetilde{\omega}\in\Omega^t$, $\widehat{Y}^{t,\omega}_\gamma(\widetilde{\omega})-\widehat{Y}^{t,\omega}_s(\widetilde{\omega})=\widehat{Y}\big(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)-\widehat{Y}\big(s,\omega\otimes_t\widetilde{\omega}\big)=Y\big(\mathfrak{s}_1(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)-Y\big(\mathfrak{s}_2(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)$, where $\mathfrak{s}_1(\widetilde{\omega}):=\gamma(\widetilde{\omega})\wedge\wp(\omega\otimes_t\widetilde{\omega})$ and $\mathfrak{s}_2(\widetilde{\omega}):=s\wedge\wp(\omega\otimes_t\widetilde{\omega})$. Let us show by two cases that

$$\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\gamma<s\}}\big|\widehat{Y}^{t,\omega}_\gamma(\widetilde{\omega})-\widehat{Y}^{t,\omega}_s(\widetilde{\omega})\big|\Big]\le\widehat{\rho}_Y(s-t). \tag{6.23}$$

If $\wp(\omega)\le t$, for any $\widetilde{\omega}\in\Omega^t$, since Lemma A.1 shows that $\wp(\omega\otimes_t\Omega^t)=\wp(\omega)$, we see that $\mathfrak{s}_1(\widetilde{\omega})=\mathfrak{s}_2(\widetilde{\omega})=\wp(\omega)$ and thus that $\big|\widehat{Y}^{t,\omega}_\gamma(\widetilde{\omega})-\widehat{Y}^{t,\omega}_s(\widetilde{\omega})\big|=0$. Otherwise, if $\wp(\omega)>t$, applying Lemma A.1 again gives that $\widehat{\gamma}:=\gamma\wedge\wp^{t,\omega}\in\mathcal{T}^t$. For any $\widetilde{\omega}\in\{\gamma<s\}$, since $\widehat{\gamma}(\widetilde{\omega})=\gamma(\widetilde{\omega})\wedge\wp(\omega\otimes_t\widetilde{\omega})=\mathfrak{s}_1(\widetilde{\omega})\ge t$ and since $\mathfrak{s}_2(\widetilde{\omega})-\mathfrak{s}_1(\widetilde{\omega})\le s-t$, (2.2) implies that

$$\begin{aligned}\big|\widehat{Y}^{t,\omega}_\gamma(\widetilde{\omega})-\widehat{Y}^{t,\omega}_s(\widetilde{\omega})\big|&\le\rho_Y\Big(\big(\mathfrak{s}_2(\widetilde{\omega})-\mathfrak{s}_1(\widetilde{\omega})\big)+\sup_{r\in[0,T]}\Big|(\omega\otimes_t\widetilde{\omega})\big(r\wedge\mathfrak{s}_1(\widetilde{\omega})\big)-(\omega\otimes_t\widetilde{\omega})\big(r\wedge\mathfrak{s}_2(\widetilde{\omega})\big)\Big|\Big)\\&\le\rho_Y\Big((s-t)+\sup_{r\in[\widehat{\gamma}(\widetilde{\omega}),\mathfrak{s}_2(\widetilde{\omega})]}\big|\widetilde{\omega}(r)-\widetilde{\omega}\big(\widehat{\gamma}(\widetilde{\omega})\big)\big|\Big)\le\rho_Y\Big((s-t)+\sup_{r\in[\widehat{\gamma}(\widetilde{\omega}),(\widehat{\gamma}(\widetilde{\omega})+s-t)\wedge T]}\big|B^t_r(\widetilde{\omega})-B^t_{\widehat{\gamma}}(\widetilde{\omega})\big|\Big).\end{aligned}$$

Then (6.23) follows from (3.5). Plugging (6.23) into (6.22), we can deduce from (4.1) and (6.16) that

$$Z_t(\omega)-Z_s(\omega)\le\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\gamma<s\}}\widehat{Y}^{t,\omega}_s+\mathbf{1}_{\{\gamma\ge s\}}Z^{t,\omega}_s-Z_s(\omega)\big]+\widehat{\rho}_Y(s-t)+\varepsilon\le\mathbb{E}_{\mathbb{P}}\big[Z^{t,\omega}_s-Z_s(\omega)\big]+\widehat{\rho}_Y(s-t)+\varepsilon\le\widehat{\phi}_{t,s}(\omega)+\widehat{\rho}_Y(s-t)+\varepsilon.$$

Letting $\varepsilon\to0$ and combining with (6.21) yield that $\big|Z_t(\omega)-Z_s(\omega)\big|\le\widehat{\phi}_{t,s}(\omega)+\widehat{\rho}_Y(s-t)$, i.e. (4.5).

As $\lim\limits_{t\nearrow s}\!\downarrow\delta_{t,s}(\omega)=\lim\limits_{s\searrow t}\!\downarrow\delta_{t,s}(\omega)=0$, we see that $\lim\limits_{t\nearrow s}\!\downarrow\widehat{\phi}_{t,s}(\omega)=\lim\limits_{s\searrow t}\!\downarrow\widehat{\phi}_{t,s}(\omega)=0$, which together with (4.1) and (4.3) shows that Z is an $\mathbf{F}-$adapted process bounded by M_Y and with all continuous paths.

3) *Finally, we show* (4.4) *for general stopping time* ν.

Fix $(t,\omega)\in[0,T]\times\Omega$, $\nu\in\mathcal{T}^t$ and $(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t$. We still take the simple notation (6.2). For any $k\in\mathbb{N}$, let us set $t^k_i:=t\vee(i2^{-k}T)$, $i=0,\cdots,2^k$ and define

$$\nu_k:=\mathbf{1}_{\{\nu=t\}}t+\sum_{i=1}^{2^k}\mathbf{1}_{\{t^k_{i-1}<\nu\le t^k_i\}}t^k_i\in\mathcal{T}^t. \tag{6.24}$$

Applying the second inequality of (6.7) with $\nu=\nu_k$ yields that $\mathbb{E}_{\mathbb{P}}[\mathcal{Y}_\gamma]\le\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\gamma\le\nu_k\}}\mathcal{Y}_\gamma+\mathbf{1}_{\{\gamma>\nu_k\}}\mathcal{Z}_{\nu_k}\big]$. Since $\lim\limits_{k\to\infty}\downarrow\nu_k=\nu$ and since

$$\text{the function } x\to\mathbf{1}_{\{x\ge a\}} \text{ is right-continuous for any } a\in\mathbb{R}, \tag{6.25}$$

letting $k\to\infty$ we can deduce from the continuity of Z by part 2), the bounded convergence theorem and (4.1) that

$$\mathbb{E}_{\mathbb{P}}[\mathcal{Y}_\gamma]\le\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\gamma\le\nu\}}\mathcal{Y}_\gamma+\mathbf{1}_{\{\gamma>\nu\}}\mathcal{Z}_\nu\big]\le\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\gamma<\nu\}}\mathcal{Y}_\gamma+\mathbf{1}_{\{\gamma\ge\nu\}}\mathcal{Z}_\nu\big]. \tag{6.26}$$

Next, let $n,k\in\mathbb{N}$ with $n<k$. We define $\gamma_n:=\mathbf{1}_{\{\gamma=t\}}t+\sum_{i=1}^{2^n}\mathbf{1}_{\{t^n_{i-1}<\gamma\le t^n_i\}}t^n_i\in\mathcal{T}^t$ and still consider ν_k defined in (6.24). Applying (6.15) with $(\mathbb{P},\gamma,\nu)=(\mathbb{P},\gamma_n,\nu_k)$ gives that

$$Z_t(\omega)\ge\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\gamma_n<\nu_k\}}\mathcal{Y}_{\gamma_n}+\mathbf{1}_{\{\gamma_n\ge\nu_k\}}\mathcal{Z}_{\nu_k}\big]. \tag{6.27}$$

Clearly, $\{\gamma_n<\nu\}\subset\{\gamma_n<\nu_k\}$. To see the reverse inclusion, we let $\omega\in\{\gamma_n<\nu_k\}$. There exist $i\in\{0,\cdots,2^n\}$ and $j\in\{1,\cdots,2^k\}$ such that $t^n_i=\gamma_n(\omega)<\nu_k(\omega)=t^k_j$. Since $\{t^n_\ell\}_{\ell=0}^{2^n}\subset\{t^k_\ell\}_{\ell=0}^{2^k}$, one has $\gamma_n(\omega)=t^n_i\le t^k_{j-1}<\nu(\omega)$. Thus $\{\gamma_n<\nu\}=\{\gamma_n<\nu_k\}$ and (6.27) becomes $Z_t(\omega)\ge\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\gamma_n<\nu\}}\mathcal{Y}_{\gamma_n}+\mathbf{1}_{\{\gamma_n\ge\nu\}}\mathcal{Z}_{\nu_k}\big]$. As $k\to\infty$, the continuity of Z by part 2), (4.1) and the bounded convergence theorem imply that

$$\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\gamma_n<\nu\}}\mathcal{Y}_{\gamma_n}+\mathbf{1}_{\{\gamma_n\ge\nu\}}\mathcal{Z}_\nu\big]=\lim_{k\to\infty}\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\gamma_n<\nu\}}\mathcal{Y}_{\gamma_n}+\mathbf{1}_{\{\gamma_n\ge\nu\}}\mathcal{Z}_{\nu_k}\big]\le Z_t(\omega).$$

Since $\lim_{n\to\infty}\downarrow\gamma_n=\gamma$, letting $n\to\infty$, we can deduce from (6.25), the continuity of Y, (4.1), the bounded convergence theorem as well as (6.26) that

$$\mathbb{E}_{\mathbb{P}}[\mathcal{Y}_\gamma]\le\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\gamma<\nu\}}\mathcal{Y}_\gamma+\mathbf{1}_{\{\gamma\ge\nu\}}\mathcal{Z}_\nu\big]=\lim_{n\to\infty}\mathbb{E}_{\mathbb{P}}\big[\mathbf{1}_{\{\gamma_n<\nu\}}\mathcal{Y}_{\gamma_n}+\mathbf{1}_{\{\gamma_n\ge\nu\}}\mathcal{Z}_\nu\big]\le Z_t(\omega).$$

Taking supremum over $(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t$ proves (4.4). □

Proof of Proposition 4.3: Fix $(Y,\wp)\in\mathfrak{S}$ and $n\in\mathbb{N}$. Since both $\widehat{Y}$ and Z are $\mathbf{F}-$adapted processes with all continuous paths by Proposition 4.2 and since

$$Z_T(\omega)=\sup_{\mathbb{P}\in\mathcal{P}_T}\mathbb{E}_{\mathbb{P}}\Big[\widehat{Y}^{T,\omega}_T\Big]=\widehat{Y}_T(\omega),\quad\forall\,\omega\in\Omega, \tag{6.28}$$

we see that

$$\nu_n:=\inf\big\{t\in[0,T]:Z_t-\widehat{Y}_t\le 1/n\big\} \tag{6.29}$$

is an $\mathbf{F}-$stopping time. Let us also fix $(t,\omega)\in[0,T]\times\Omega$.

1) *Given $\zeta\in\mathcal{T}$, let us first show that*

$$Z_{\zeta\wedge t}(\omega)\ge\overline{\mathscr{E}}_t[Z_\zeta](\omega). \tag{6.30}$$

If $\widehat{t}:=\zeta(\omega)\le t$, applying Lemma A.1 with $(t,s,\tau)=(0,t,\zeta)$ shows that $\zeta(\omega\otimes_t\Omega^t)\equiv\widehat{t}$. Since $Z_{\widehat{t}}\in\mathcal{F}_{\widehat{t}}\subset\mathcal{F}_t$ by Proposition 4.2, using (2.6) with $(t,s,\eta)=\big(0,t,Z_{\widehat{t}}\big)$ shows that

$$(Z_\zeta)^{t,\omega}(\widetilde{\omega})=Z_\zeta(\omega\otimes_t\widetilde{\omega})=Z\big(\widehat{t},\omega\otimes_t\widetilde{\omega}\big)=Z\big(\widehat{t},\omega\big)=Z\big(\zeta(\omega)\wedge t,\omega\big),\quad\forall\,\widetilde{\omega}\in\Omega^t. \tag{6.31}$$

It follows that $\overline{\mathscr{E}}_t[Z_\zeta](\omega)=\sup\limits_{\mathbb{P}\in\mathcal{P}_t}\mathbb{E}_{\mathbb{P}}\big[(Z_\zeta)^{t,\omega}\big]=Z_{\zeta\wedge t}(\omega)$.

On the other hand, if $\zeta(\omega)>t$, as $\zeta^{t,\omega}\in\mathcal{T}^t$ by Lemma A.1, applying (4.4) with $\gamma=\nu=\zeta^{t,\omega}$ yields that

$$Z_{\zeta\wedge t}(\omega)=Z_t(\omega)=\sup_{(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\gamma<\zeta^{t,\omega}\}}\widehat{Y}^{t,\omega}_\gamma+\mathbf{1}_{\{\gamma\ge\zeta^{t,\omega}\}}Z^{t,\omega}_{\zeta^{t,\omega}}\Big]\ge\sup_{\mathbb{P}\in\mathcal{P}_t}\mathbb{E}_{\mathbb{P}}\big[(Z_\zeta)^{t,\omega}\big]=\overline{\mathscr{E}}_t[Z_\zeta](\omega).$$

2) *Let $\zeta\in\mathcal{T}$. We next show that $Z_{\nu_n\wedge\zeta\wedge t}(\omega)\le\overline{\mathscr{E}}_t[Z_{\nu_n\wedge\zeta}](\omega)$.*

If $\nu_n(\omega)\wedge\zeta(\omega)\le t$, using similar arguments that lead to (6.31) yields that $(Z_{\nu_n\wedge\zeta})^{t,\omega}(\widetilde{\omega})=Z\big(\nu_n(\omega)\wedge\zeta(\omega)\wedge t,\omega\big)$, $\forall\,\widetilde{\omega}\in\Omega^t$ and thus $\overline{\mathscr{E}}_t[Z_{\nu_n\wedge\zeta}](\omega)=Z_{\nu_n\wedge\zeta\wedge t}(\omega)$.

On the other hand, suppose that $\nu_n(\omega)\wedge\zeta(\omega)>t$. We see from Lemma A.1 again that $\zeta_n:=(\nu_n\wedge\zeta)^{t,\omega}\in\mathcal{T}^t$. Let $\varepsilon>0$. Applying (4.4) with $\nu=\zeta_n$, one can find a pair $(\mathbb{P}_\varepsilon,\gamma_\varepsilon)=(\mathbb{P}^n_\varepsilon,\gamma^n_\varepsilon)\in\mathcal{P}_t\times\mathcal{T}^t$ such that

$$Z_t(\omega)=\sup_{(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t}\mathbb{E}_{\mathbb{P}}\Big[\mathbf{1}_{\{\gamma<\zeta_n\}}\widehat{Y}^{t,\omega}_\gamma+\mathbf{1}_{\{\gamma\ge\zeta_n\}}Z^{t,\omega}_{\zeta_n}\Big]\le\mathbb{E}_{\mathbb{P}_\varepsilon}\Big[\mathbf{1}_{\{\gamma_\varepsilon<\zeta_n\}}\widehat{Y}^{t,\omega}_{\gamma_\varepsilon}+\mathbf{1}_{\{\gamma_\varepsilon\ge\zeta_n\}}Z^{t,\omega}_{\zeta_n}\Big]+\varepsilon. \tag{6.32}$$

For any $\widetilde{\omega}\in\{\gamma_\varepsilon<\zeta_n\}$, since $\gamma_\varepsilon(\widetilde{\omega})<\zeta_n(\widetilde{\omega})=(\nu_n\wedge\zeta)(\omega\otimes_t\widetilde{\omega})\le\nu_n(\omega\otimes_t\widetilde{\omega})$, the definition of ζ_n shows that $\frac{1}{n}<Z\big(\gamma_\varepsilon(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)-\widehat{Y}\big(\gamma_\varepsilon(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)=Z^{t,\omega}_{\gamma_\varepsilon}(\widetilde{\omega})-\widehat{Y}^{t,\omega}_{\gamma_\varepsilon}(\widetilde{\omega})$. It follows from (6.32) that

$$Z_t(\omega)\le\mathbb{E}_{\mathbb{P}_\varepsilon}\Big[\mathbf{1}_{\{\gamma_\varepsilon<\zeta_n\}}\widehat{Y}^{t,\omega}_{\gamma_\varepsilon}+\mathbf{1}_{\{\gamma_\varepsilon\ge\zeta_n\}}Z^{t,\omega}_{\zeta_n}\Big]+\varepsilon\le\mathbb{E}_{\mathbb{P}_\varepsilon}\Big[Z^{t,\omega}_{\gamma_\varepsilon\wedge\zeta_n}-\frac{1}{n}\mathbf{1}_{\{\gamma_\varepsilon<\zeta_n\}}\Big]+\varepsilon. \tag{6.33}$$

Since $\gamma_\varepsilon(\Pi^0_t)\in\mathcal{T}_t$ by Lemma A.1 of [6], applying (6.30) with $\zeta=\gamma_\varepsilon(\Pi^0_t)\wedge\nu_n\wedge\zeta$ yields that

$$Z_t(\omega)=Z_{\gamma_\varepsilon(\Pi^0_t)\wedge\nu_n\wedge\zeta\wedge t}(\omega)\ge\overline{\mathscr{E}}_t[Z_{\gamma_\varepsilon(\Pi^0_t)\wedge\nu_n\wedge\zeta}](\omega)\ge\mathbb{E}_{\mathbb{P}_\varepsilon}\Big[\big(Z_{\gamma_\varepsilon(\Pi^0_t)\wedge\nu_n\wedge\zeta}\big)^{t,\omega}\Big]=\mathbb{E}_{\mathbb{P}_\varepsilon}\Big[Z^{t,\omega}_{\gamma_\varepsilon\wedge\zeta_n}\Big], \tag{6.34}$$

where we used the fact that for any $\widetilde{\omega}\in\Omega^t$

$$\big(Z_{\gamma_\varepsilon(\Pi^0_t)\wedge\nu_n\wedge\zeta}\big)^{t,\omega}(\widetilde{\omega})=Z\big(\gamma_\varepsilon\big(\Pi^0_t(\omega\otimes_t\widetilde{\omega})\big)\wedge\nu_n(\omega\otimes_t\widetilde{\omega})\wedge\zeta(\omega\otimes_t\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)=Z\big(\gamma_\varepsilon(\widetilde{\omega})\wedge\zeta_n(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)=Z^{t,\omega}_{\gamma_\varepsilon\wedge\zeta_n}(\widetilde{\omega}).$$

Putting (6.33) and (6.34) together shows that $\mathbb{P}_\varepsilon\{\gamma_\varepsilon<\zeta_n\}\le n\varepsilon$. Then we can deduce from (6.32) and (4.1) that

$$\begin{aligned}Z_{\nu_n\wedge\zeta\wedge t}(\omega)&=Z_t(\omega)\le\mathbb{E}_{\mathbb{P}_\varepsilon}\Big[\mathbf{1}_{\{\gamma_\varepsilon<\zeta_n\}}(\widehat{Y}^{t,\omega}_{\gamma_\varepsilon}-Z^{t,\omega}_{\zeta_n})+Z^{t,\omega}_{\zeta_n}\Big]+\varepsilon\le2M_Y\mathbb{P}_\varepsilon\{\gamma_\varepsilon<\zeta_n\}+\mathbb{E}_{\mathbb{P}_\varepsilon}[Z^{t,\omega}_{\zeta_n}]+\varepsilon\\&\le\mathbb{E}_{\mathbb{P}_\varepsilon}\big[(Z_{\nu_n\wedge\zeta})^{t,\omega}\big]+(1+2nM_Y)\varepsilon\le\overline{\mathscr{E}}_t[Z_{\nu_n\wedge\zeta}](\omega)+(1+2nM_Y)\varepsilon.\end{aligned}$$

Letting $\varepsilon\to0$ yields that $Z_{\nu_n\wedge\zeta\wedge t}(\omega)\le\overline{\mathscr{E}}_t[Z_{\nu_n\wedge\zeta}](\omega)$. □

Proof of Theorem 4.1: Fix $(Y,\wp)\in\mathfrak{S}$. Since both $\widehat{Y}$ and Z are $\mathbf{F}-$adapted processes with all continuous paths by Proposition 4.2, we see from (4.1) and (6.28) that $\widehat{\nu}:=\inf\big\{t\in[0,T]:Z_t=\widehat{Y}_t\big\}$ is an $\mathbf{F}-$stopping time. For any $n\in\mathbb{N}$, let ν_n be the $\mathbf{F}-$stopping time defined in (6.29). Since Z is an $\overline{\mathscr{E}}-$martingale over $[0,\nu_n]$ by Proposition 4.3, one can find a $\mathbb{P}_n\in\mathcal{P}$ satisfying (1.4). By (P1), $\{\mathbb{P}_n\}_{n=2}^\infty$ has a weakly convergent subsequence $\{\mathbb{P}_{m_j}\}_{j\in\mathbb{N}}$ with limit $\widehat{\mathbb{P}}\in\mathcal{P}$.

When $m_j\ge n$, (so $\nu_n\le\nu_{m_j}$), applying Lemma A.3 with $(\mathbb{P},\tau,\gamma)=(\mathbb{P}_{m_j},\nu_n,\nu_{m_j})$, we see from (1.4) that

$$Z_0\le\mathbb{E}_{\mathbb{P}_{m_j}}\big[Z_{\nu_{m_j}}\big]+2^{-m_j}\le\mathbb{E}_{\mathbb{P}_{m_j}}\big[Z_{\nu_n}\big]+2^{-m_j}. \tag{6.35}$$

1) *Before sending j to ∞ in order to approximate the distribution $\widehat{\mathbb{P}}$ in* (6.35)*, we need to approach $\{\nu_n\}_{n\in\mathbb{N}}$ by a sequence $\big\{\widehat{\theta}_n\big\}_{n\in\mathbb{N}}$ of Lipschitz continuous random variables.*

Fix integer $n\ge2$. There exists a $\lambda_n>0$ such that $\rho_Y(x)\vee\widehat{\rho}_Y(x)\le\frac{1}{2n(n+1)}$, $\forall\,x\in[0,\lambda_n]$. Let $\omega\in\Omega$, set $\delta_n(\omega):=\dfrac{\lambda_n}{2(1+\kappa_\wp)}\wedge\dfrac{(\phi^\omega_T)^{-1}(\lambda_n/2)}{\kappa_\wp}$ with $(\phi^\omega_T)^{-1}(x):=\inf\{y>0:\phi^\omega_T(y)=x\}$, $\forall\,x>0$, and let $\omega'\in\overline{O}_{\delta_n(\omega)}(\omega)$. Given $t\in[0,T]$, set $s:=\wp(\omega)\wedge t$ and $s':=\wp(\omega')\wedge t$. By (2.5), $|s-s'|\le\big|\wp(\omega)-\wp(\omega')\big|\le\kappa_\wp\|\omega-\omega'\|_{0,T}$. Then (2.2) implies that

$$\begin{aligned}\big|\widehat{Y}(t,\omega)-\widehat{Y}(t,\omega')\big|&=|Y(s,\omega)-Y(s',\omega')|\le\rho_Y\Big(|s-s'|+\sup_{r\in[0,T]}\big|\omega(r\wedge s)-\omega'(r\wedge s')\big|\Big)\\&\le\rho_Y\Big(\kappa_\wp\|\omega-\omega'\|_{0,T}+\sup_{r\in[0,T]}|\omega(r\wedge s)-\omega(r\wedge s')|+\sup_{r\in[0,T]}\big|\omega(r\wedge s')-\omega'(r\wedge s')\big|\Big)\\&\le\rho_Y\Big((1+\kappa_\wp)\|\omega-\omega'\|_{0,T}+\phi^\omega_T\big(|s'-s|\big)\Big)\le\rho_Y\Big((1+\kappa_\wp)\|\omega-\omega'\|_{0,T}+\phi^\omega_T\big(\kappa_\wp\|\omega-\omega'\|_{0,T}\big)\Big)\le\frac{1}{2n(n+1)}.\end{aligned} \tag{6.36}$$

Taking $t=\nu_n(\omega)$, we see from (4.2) that

$$\begin{aligned}\big|(Z-\widehat{Y})(\nu_n(\omega),\omega)-(Z-\widehat{Y})(\nu_n(\omega),\omega')\big|&\le\big|Z(\nu_n(\omega),\omega)-Z(\nu_n(\omega),\omega')\big|+\big|\widehat{Y}(\nu_n(\omega),\omega)-\widehat{Y}(\nu_n(\omega),\omega')\big|\\&\le\widehat{\rho}_Y\Big((1+\kappa_\wp)\|\omega-\omega'\|_{0,\nu_n(\omega)}+\phi^\omega_{\nu_n(\omega)}\big(\kappa_\wp\|\omega-\omega'\|_{0,\nu_n(\omega)}\big)\Big)+\frac{1}{2n(n+1)}\\&\le\widehat{\rho}_Y\Big((1+\kappa_\wp)\|\omega-\omega'\|_{0,T}+\phi^\omega_T\big(\kappa_\wp\|\omega-\omega'\|_{0,T}\big)\Big)+\frac{1}{2n(n+1)}\le\frac{1}{n(n+1)}<\frac{1}{(n-1)n}.\end{aligned}$$

As the continuity of $Z-\widehat{Y}$ shows that

$$\big(Z-\widehat{Y}\big)(\nu_n(\omega),\omega)\le\frac{1}{n}, \tag{6.37}$$

it follows that $(Z-\widehat{Y})(\nu_n(\omega),\omega')\le \frac{1}{n}+\frac{1}{(n-1)n}=\frac{1}{n-1}$, so $\nu_{n-1}(\omega')\le\nu_n(\omega)$. Analogously, taking $t=\nu_{n+1}(\omega')$ in (6.36) yields that

$$\begin{aligned}&\big|(Z-\widehat{Y})(\nu_{n+1}(\omega'),\omega)-(Z-\widehat{Y})(\nu_{n+1}(\omega'),\omega')\big|\le\big|Z(\nu_{n+1}(\omega'),\omega)-Z(\nu_{n+1}(\omega'),\omega')\big|+\big|\widehat{Y}(\nu_{n+1}(\omega'),\omega)-\widehat{Y}(\nu_{n+1}(\omega'),\omega')\big|\\&\quad\le\widehat{\rho}_Y\Big((1+\kappa_\wp)\|\omega-\omega'\|_{0,T}+\phi^\omega_T\big(\kappa_\wp\|\omega-\omega'\|_{0,T}\big)\Big)+\frac{1}{2n(n+1)}\le\frac{1}{n(n+1)},\end{aligned}$$

and that $(Z-\widehat{Y})(\nu_{n+1}(\omega'),\omega)<(Z-\widehat{Y})(\nu_{n+1}(\omega'),\omega')+\frac{1}{n(n+1)}\le\frac{1}{n+1}+\frac{1}{n(n+1)}=\frac{1}{n}$, which shows that $\nu_n(\omega)\le\nu_{n+1}(\omega')$.

Now, we can apply Lemma A.4 with $(\Omega_0,\underline{\theta},\theta,\overline{\theta},I,\delta(\omega),\varepsilon)=(\Omega,\nu_{n-1},\nu_n,\nu_{n+1},[0,T],\delta_n(\omega),2^{-n})$ to find an open subset $\widehat{\Omega}_n$ of Ω and a Lipschitz continuous random variable $\widehat{\theta}_n:\Omega\to[0,T]$ such that

$$\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{P}\big(\widehat{\Omega}^c_n\big)\le 2^{-n},\quad \nu_{n-1}-2^{-n}<\widehat{\theta}_n<\nu_{n+1}+2^{-n}\ \text{ on }\ \widehat{\Omega}_n. \tag{6.38}$$

2) *Next, let us estimate the expected difference* $\mathbb{E}_{\mathbb{P}_{m_j}}\big[\big|Z_{\widehat{\theta}_n}-Z_{\nu_n}\big|\big]$.

Given $\omega\in\widehat{\Omega}_{n-1}\cap\widehat{\Omega}_{n+1}$, as $\widehat{\theta}_{n-1}-2^{-n+1}<\nu_n<\widehat{\theta}_{n+1}+2^{-n-1}$, $t:=\widehat{\theta}_n(\omega)\wedge\nu_n(\omega)$ and $s:=\widehat{\theta}_n(\omega)\vee\nu_n(\omega)$ satisfy

$$\begin{aligned}s-t&=\big|\nu_n(\omega)-\widehat{\theta}_n(\omega)\big|<(\widehat{\theta}_{n-1}-\widehat{\theta}_n-2^{-n+1})^-\vee(\widehat{\theta}_{n+1}-\widehat{\theta}_n+2^{-n-1})^+\le|\widehat{\theta}_{n-1}-\widehat{\theta}_n-2^{-n+1}|\vee|\widehat{\theta}_{n+1}-\widehat{\theta}_n+2^{-n-1}|\\&\le|\widehat{\theta}_{n-1}(\omega)-\widehat{\theta}_n(\omega)|+|\widehat{\theta}_{n+1}(\omega)-\widehat{\theta}_n(\omega)|+2^{-n+1}:=\widehat{\delta}_n(\omega).\end{aligned}\tag{6.39}$$

Set $\phi_n(\omega):=(1+\kappa_\wp)\Big(\big(\widehat{\delta}_n(\omega)\big)^{\frac{q_1}{2}}+\phi^\omega_T\big(\widehat{\delta}_n(\omega)\big)\Big)$. Then (4.5) shows that

$$\big|Z_{\widehat{\theta}_n}(\omega)-Z_{\nu_n}(\omega)\big|=\big|Z(t,\omega)-Z(s,\omega)\big|\le 2C_\varrho M_Y\Big(\big(\widehat{\delta}_n(\omega)\big)^{\frac{q_1}{2}}\vee\big(\widehat{\delta}_n(\omega)\big)^{q_2-\frac{q_1}{2}}\Big)+\widehat{\rho}_Y\big(\widehat{\delta}_n(\omega)\big)+\widehat{\rho}_Y\big(\phi_n(\omega)\big)\vee\widehat{\overline{\rho}}_Y\big(\phi_n(\omega)\big):=\xi_n(\omega).$$

Let $j\in\mathbb{N}$ with $m_j\ge n$. We see from (6.35), (4.1) and (6.38) that

$$\begin{aligned}Z_0-2^{-m_j}&\le\mathbb{E}_{\mathbb{P}_{m_j}}\big[Z_{\widehat{\theta}_n}\big]+\mathbb{E}_{\mathbb{P}_{m_j}}\big[\big|Z_{\widehat{\theta}_n}-Z_{\nu_n}\big|\big]\le\mathbb{E}_{\mathbb{P}_{m_j}}\big[Z_{\widehat{\theta}_n}+\mathbf{1}_{\widehat{\Omega}_{n-1}\cap\widehat{\Omega}_{n+1}}(\xi_n\wedge 2M_Y)\big]+2M_Y\mathbb{P}_{m_j}\big(\widehat{\Omega}^c_{n-1}\cup\widehat{\Omega}^c_{n+1}\big)\\&\le\mathbb{E}_{\mathbb{P}_{m_j}}\big[Z_{\widehat{\theta}_n}+(\xi_n\wedge 2M_Y)\big]+5M_Y2^{-n}.\end{aligned}\tag{6.40}$$

The random variables $\widehat{\theta}_{n-1},\widehat{\theta}_n,\widehat{\theta}_{n+1}$ are Lipschitz continuous on Ω, so is $\widehat{\delta}_n$. Then one can deduce that

$$\omega\to\phi^\omega_T\big(\widehat{\delta}_n(\omega)\big)\ \text{is a continuous random variable on }\Omega,\tag{6.41*}$$

which together with the Lipschitz continuity of $\widehat{\delta}_n$ implies that ϕ_n and thus ξ_n are continuous random variables on Ω. Moreover, the Lipschitz continuity of random variable $\widehat{\theta}_n$ and the continuity of process Z implies that

$$Z_{\widehat{\theta}_n}\ \text{ is also a continuous random variable on }\Omega.\tag{6.42*}$$

Letting $j\to\infty$ in (6.40), we see from the continuity of random variables $Z_{\widehat{\theta}_n}$ and ξ_n that

$$Z_0\le\mathbb{E}_{\widehat{\mathbb{P}}}\big[Z_{\widehat{\theta}_n}+(\xi_n\wedge 2M_Y)\big]+5M_Y2^{-n},\quad\forall\, n\ge 2.\tag{6.43}$$

3) *Finally, we use the convergence of* $\widehat{\theta}_n$ *to* $\widehat{\nu}$ *and the continuity of* Z *to derive the* $\overline{\mathscr{E}}-$*martingality of* Z *over* $[0,\widehat{\nu}]$.

Set $\widehat{\nu}':=\lim\limits_{n\to\infty}\uparrow\nu_n\le\widehat{\nu}$. The continuity of $Z-\widehat{Y}$, (6.37) and (4.1) imply that $Z_{\widehat{\nu}'}-\widehat{Y}_{\widehat{\nu}'}=0$, thus $\widehat{\nu}=\widehat{\nu}'=\lim\limits_{n\to\infty}\uparrow\nu_n$. Then we can deduce from (6.38) that $\lim\limits_{n\to\infty}\widehat{\theta}_n(\omega)=\widehat{\nu}(\omega)$, $\forall\,\omega\in\bigcup\limits_{n=3}^{\infty}\bigcap\limits_{k\ge n}\widehat{\Omega}_k$. As $\sum\limits_{n=3}^{\infty}\widehat{\mathbb{P}}\big(\widehat{\Omega}^c_n\big)\le\sum\limits_{n=3}^{\infty}\sup\limits_{\mathbb{P}\in\mathcal{P}}\mathbb{P}\big(\widehat{\Omega}^c_n\big)\le\frac{1}{4}$, the Borel-Cantelli Lemma implies that $\widehat{\mathbb{P}}\Big(\bigcup\limits_{n=3}^{\infty}\bigcap\limits_{k\ge n}\widehat{\Omega}_k\Big)=1$. So

$$\lim_{n\to\infty}\widehat{\theta}_n=\widehat{\nu},\quad\widehat{\mathbb{P}}-a.s.\tag{6.44}$$

It follows that $\lim\limits_{n\to\infty}\widehat{\delta}_n=0$, $\widehat{\mathbb{P}}-$a.s. and thus $\lim\limits_{n\to\infty}\xi_n=0$, $\widehat{\mathbb{P}}-$a.s. Eventually, letting $n\to\infty$ in (6.43), we can deduce from the continuity of process Z, $\widehat{Y}$ and the bounded dominated convergence theorem that

$$Z_0\le\mathbb{E}_{\widehat{\mathbb{P}}}\big[Z_{\widehat{\nu}}\big]\le\overline{\mathscr{E}}_0\big[Z_{\widehat{\nu}}\big]=\overline{\mathscr{E}}_0\big[\widehat{Y}_{\widehat{\nu}}\big]\le\sup_{(\mathbb{P},\gamma)\in\mathcal{P}\times\mathcal{T}}\mathbb{E}_{\mathbb{P}}\big[\widehat{Y}_\gamma\big]=Z_0.$$

Hence, $Z_0=\overline{\mathscr{E}}_0\big[Z_{\widehat{\nu}}\big]=\mathbb{E}_{\widehat{\mathbb{P}}}\big[Z_{\widehat{\nu}}\big]=\mathbb{E}_{\widehat{\mathbb{P}}}\big[\widehat{Y}_{\widehat{\nu}}\big]$.

Next, let $\zeta\in\mathcal{T}$. For any $\mathbb{P}\in\mathcal{P}$, we see from Lemma A.3 that

$$Z_0=\mathbb{E}_{\mathbb{P}}[Z_0]\ge\mathbb{E}_{\mathbb{P}}\big[Z_{\widehat{\nu}\wedge\zeta}\big]\ge\mathbb{E}_{\mathbb{P}}\big[Z_{\widehat{\nu}}\big].\tag{6.45}$$

Taking supremum over $\mathbb{P}\in\mathcal{P}$ yields that $Z_0\ge\overline{\mathscr{E}}_0\big[Z_{\widehat{\nu}\wedge\zeta}\big]\ge\overline{\mathscr{E}}_0\big[Z_{\widehat{\nu}}\big]=Z_0$. In particular, taking $\mathbb{P}=\widehat{\mathbb{P}}$ in (6.45) shows that $Z_0\ge\mathbb{E}_{\widehat{\mathbb{P}}}\big[Z_{\widehat{\nu}\wedge\zeta}\big]\ge\mathbb{E}_{\widehat{\mathbb{P}}}\big[Z_{\widehat{\nu}}\big]=Z_0$. □

6.3 Proofs of results in Section 5

Proof of Proposition 5.1: Set $\mathfrak{n}_0:=1+\lfloor\mathscr{X}_0^{-1}\rfloor>\mathscr{X}_0^{-1}$. Given $k\in\mathbb{N}\cup\{0\}$, since $\mathscr{X}$ is an $\mathbf{F}-$adapted process with all continuous paths and since $\mathscr{X}_0>\frac{1}{\mathfrak{n}_0}\ge\frac{1}{k+\mathfrak{n}_0}$, we see that $\widehat{\tau}_k:=\inf\big\{t\in[0,T]:\,\mathscr{X}_t\le\frac{1}{k+\mathfrak{n}_0}\big\}\wedge T$ is an $\mathbf{F}-$stopping time satisfying $0<\widehat{\tau}_k(\omega)\le\tau_0(\omega),\ \forall\,\omega\in\Omega$. In particular, if $\{t\in[0,T]:\,\mathscr{X}_t(\omega')\le 0\}$ is not empty for some $\omega'\in\Omega$, then $\widehat{\tau}_k(\omega')<\tau_0(\omega')$. Let $\{\delta_k\}_{k\in\mathbb{N}}$ be a sequence decreasing to 0 such that $\rho_{\mathscr{X}}(\delta_k)\le\frac{1}{(k+\mathfrak{n}_0)(k+\mathfrak{n}_0+1)},\ \forall\,k\in\mathbb{N}$.

a) *First, we construct an auxiliary increasing sequence $\{\vartheta_\ell\}_{\ell\in\mathbb{N}}$ of Lipschitz continuous stopping times.*

Fix $k\in\mathbb{N}$. For $i=k-1,\,k$, let $\omega,\omega'\in\Omega$ with $\|\omega'-\omega\|_{0,\widehat{\tau}_{i+1}(\omega)}\le\delta_k$, (3.1) shows that

$$\big|\mathscr{X}\big(\widehat{\tau}_{i+1}(\omega),\omega'\big)-\mathscr{X}\big(\widehat{\tau}_{i+1}(\omega),\omega\big)\big|\le\rho_{\mathscr{X}}\big(\|\omega'-\omega\|_{0,\widehat{\tau}_{i+1}(\omega)}\big)\le\rho_{\mathscr{X}}(\delta_k)\le\frac{1}{(k+\mathfrak{n}_0)(k+\mathfrak{n}_0+1)}.$$

If $\mathscr{X}_t(\omega)>\frac{1}{i+\mathfrak{n}_0+1}$ for all $t\in[0,T]$, then $\widehat{\tau}_{i+1}(\omega)=T\ge\widehat{\tau}_i(\omega')$. On the other hand, if the set $\big\{t\in[0,T]:\,\mathscr{X}_t(\omega)\le\frac{1}{i+\mathfrak{n}_0+1}\big\}$is not empty, the continuity of $\mathscr{X}$ imply that $\mathscr{X}\big(\widehat{\tau}_{i+1}(\omega),\omega\big)=\frac{1}{i+\mathfrak{n}_0+1}$, it follows that $\mathscr{X}\big(\widehat{\tau}_{i+1}(\omega),\omega'\big)\le\frac{1}{i+\mathfrak{n}_0+1}+\frac{1}{(i+\mathfrak{n}_0)(i+\mathfrak{n}_0+1)}=\frac{1}{i+\mathfrak{n}_0}$, so one still has $\widehat{\tau}_i(\omega')\le\widehat{\tau}_{i+1}(\omega)$. Then we can apply Lemma A.6 with $(\theta_1,\theta_2,\theta_3,\delta,\kappa)=\big(\widehat{\tau}_{k-1},\widehat{\tau}_k,\widehat{\tau}_{k+1},\delta_k,2T/\delta_k\big)$ to find a $\widehat{\wp}_k\in\mathcal{T}$ such that

$$\widehat{\tau}_{k-1}(\omega)\le\widehat{\wp}_k(\omega)\le\widehat{\tau}_{k+1}(\omega)\le\tau_0(\omega),\quad\ \forall\,\omega\in\Omega,\tag{6.46}$$

(the last inequality is strict if the set $\{t\in[0,T]:\,\mathscr{X}_t(\omega)\le 0\}$ is not empty) and that given $\omega_1,\omega_2\in\Omega$,

$$\big|\widehat{\wp}_k(\omega_1)-\widehat{\wp}_k(\omega_2)\big|\le 2T\delta_k^{-1}\|\omega_1-\omega_2\|_{0,t_0}\tag{6.47}$$

holds for any $t_0\in\big[\widehat{b}_k,T\big]\cup\big\{t\in[\widehat{a}_k,\widehat{b}_k):\,t\ge\widehat{a}_k+2T\delta_k^{-1}\|\omega_1-\omega_2\|_{0,t}\big\}$, where $\widehat{a}_k:=\widehat{\wp}_k(\omega_1)\wedge\widehat{\wp}_k(\omega_2)$ and $\widehat{b}_k:=\widehat{\wp}_k(\omega_1)\vee\widehat{\wp}_k(\omega_2)$.

Let $\ell\in\mathbb{N}$. We define and $\mathbf{F}-$stopping time $\vartheta_\ell:=\max\limits_{k=1,\cdots,\ell}\widehat{\wp}_k$. Let $\omega_1,\omega_2\in\mathbb{N}$ and set $\mathfrak{a}_\ell:=\vartheta_\ell(\omega_1)\wedge\vartheta_\ell(\omega_2)$, $\mathfrak{b}_\ell:=\vartheta_\ell(\omega_1)\vee\vartheta_\ell(\omega_2)$. To see that

$$\big|\vartheta_\ell(\omega_1)-\vartheta_\ell(\omega_2)\big|\le 2T\delta_\ell^{-1}\|\omega_1-\omega_2\|_{0,t_0}\tag{6.48}$$

holds for any $t_0\in[\mathfrak{b}_\ell,T]\cup\big\{t\in[\mathfrak{a}_\ell,\mathfrak{b}_\ell):\,t\ge\mathfrak{a}_\ell+2T\delta_\ell^{-1}\|\omega_1-\omega_2\|_{0,t}\big\}$, we first let $t_0\in[\mathfrak{b}_\ell,T]$. For any $k=1,\cdots,\ell$, since $\widehat{b}_k\le\vartheta_\ell(\omega_1)\vee\vartheta_\ell(\omega_2)=\mathfrak{b}_\ell\le t_0$, applying (6.47) yields that

$$\big|\widehat{\wp}_k(\omega_1)-\widehat{\wp}_k(\omega_2)\big|\le 2T\delta_k^{-1}\|\omega_1-\omega_2\|_{0,t_0}\le 2T\delta_\ell^{-1}\|\omega_1-\omega_2\|_{0,t_0}.\tag{6.49}$$

It follows that $\widehat{\wp}_k(\omega_1)\le\widehat{\wp}_k(\omega_2)+2T\delta_\ell^{-1}\|\omega_1-\omega_2\|_{0,t_0}\le\vartheta_\ell(\omega_2)+2T\delta_\ell^{-1}\|\omega_1-\omega_2\|_{0,t_0}$. Taking maximum over $k=1,\cdots,\ell$ shows that $\vartheta_\ell(\omega_1)\le\vartheta_\ell(\omega_2)+2T\delta_\ell^{-1}\|\omega_1-\omega_2\|_{0,t_0}$. Then exchanging the roles of ω_1 and ω_2 yields (6.48).

We next suppose that the set $\big\{t\in[\mathfrak{a}_\ell,\mathfrak{b}_\ell):\,t\ge\mathfrak{a}_\ell+2T\delta_\ell^{-1}\|\omega_1-\omega_2\|_{0,t}\big\}$ is not empty and contains t_0. Given $k=1,\cdots,\ell$, since $t_0\in[\mathfrak{a}_\ell,\mathfrak{b}_\ell)\subset\big[\widehat{a}_k,T\big]$ and since

$$\widehat{\wp}_k(\omega_1)\wedge\widehat{\wp}_k(\omega_2)+2T\delta_k^{-1}\|\omega_1-\omega_2\|_{0,t_0}\le\vartheta_\ell(\omega_1)\wedge\vartheta_\ell(\omega_2)+2T\delta_\ell^{-1}\|\omega_1-\omega_2\|_{0,t_0}\le t_0,$$

applying (6.47) yields (6.49) and thus leads to (6.48) again.

Now, fix $n\in\mathbb{N}$. We set $\ell:=\lceil\log_2(n+2)\rceil\ge 2$, $j:=n+2-2^{\ell-1}$ and define $\wp_n:=(\vartheta_{\ell-1}+j2^{1-\ell}T)\wedge\vartheta_\ell\in\mathcal{T}$.

b) *In this step, we show that $\wp_n$'s is the increasing sequence of Lipschitz continuous stopping times in quest such that the increment $\wp_{n+1}-\wp_n$ is bounded by $\frac{2T}{n+3}$.*

Since $\ell-1<\log_2(n+2)\le\ell$, we see that $1\le j\le 2^{\ell-1}$. If $j<2^{\ell-1}$, as $n+2=2^{\ell-1}+j\le 2^\ell-1$, one has $\ell=\lceil\log_2(n+2)\rceil\le\lceil\log_2(n+3)\rceil\le\ell$, so $\lceil\log_2(n+3)\rceil=\ell$. Then (2.5) implies that

$$0\le\wp_{n+1}(\omega)-\wp_n(\omega)=\big(\vartheta_{\ell-1}(\omega)+(j+1)2^{1-\ell}T\big)\wedge\vartheta_\ell(\omega)-\big(\vartheta_{\ell-1}(\omega)+j2^{1-\ell}T\big)\wedge\vartheta_\ell(\omega)\le 2^{1-\ell}T\le\frac{2T}{n+3},\quad\forall\,\omega\in\Omega.$$

On the other hand, if $j=2^{\ell-1}$, i.e. $n+2=2^\ell$, then $\wp_n=(\vartheta_{\ell-1}+T)\wedge\vartheta_\ell=\vartheta_\ell$ and $\lceil\log_2(n+3)\rceil=\lceil\log_2(2^\ell+1)\rceil=\ell+1$. Applying (2.5) again yields that

$$0\le\wp_{n+1}(\omega)-\wp_n(\omega)=\big(\vartheta_\ell(\omega)+2^{-\ell}T\big)\wedge\vartheta_{\ell+1}(\omega)-\vartheta_\ell(\omega)\wedge\vartheta_{\ell+1}(\omega)\le 2^{-\ell}T=\frac{T}{n+2}<\frac{2T}{n+3},\quad\forall\,\omega\in\Omega.$$

Since $\widehat{\tau}_{\ell-2}=\inf\big\{t\in[0,T]:\mathscr{X}_t\le\frac{1}{\ell-2+\mathfrak{n}_0}\big\}\wedge T=\inf\big\{t\in[0,T]:\mathscr{X}_t\le(\lceil\log_2(n+2)\rceil+\lfloor\mathscr{X}_0^{-1}\rfloor-1)^{-1}\big\}\wedge T=\tau_n$ by (3.2), we can deduce from (6.46) that

$$\tau_n(\omega)=\widehat{\tau}_{\ell-2}(\omega)\le\widehat{\wp}_{\ell-1}(\omega)\le\vartheta_{\ell-1}(\omega)\le(\vartheta_{\ell-1}(\omega)+j2^{1-\ell}T)\wedge\vartheta_\ell(\omega)=\wp_n(\omega)\le\vartheta_\ell(\omega)=\max_{i=1,\cdots,\ell}\widehat{\wp}_k(\omega)\le\tau_0(\omega),\ \ \forall\,\omega\in\Omega,$$

where the last inequality is strict if the set $\{t\in[0,T]:\mathscr{X}_t(\omega)\le 0\}$ is not empty.

c) *It remains to show the Lipschitz continuity of $\wp_n$.*

Set $\kappa_n:=2T\delta_\ell^{-1}=2T\big(\delta_{\lceil\log_2(n+2)\rceil}\big)^{-1}$, which is increasing in n and converges to ∞. Let $\omega_1,\omega_2\in\mathbb{N}$ and set $a_n:=\wp_n(\omega_1)\wedge\wp_n(\omega_2)$. We assume without loss of generality that $a_n=\wp_n(\omega_1)\le\wp_n(\omega_2)$ and discuss by two cases:
i) When $\wp_n(\omega_1)=\vartheta_{\ell-1}(\omega_1)+j2^{1-\ell}T$, one has

$$\wp_n(\omega_2)-\wp_n(\omega_1)=\wp_n(\omega_2)-\vartheta_{\ell-1}(\omega_1)-j2^{1-\ell}T\le\vartheta_{\ell-1}(\omega_2)-\vartheta_{\ell-1}(\omega_1).\tag{6.50}$$

Applying (6.48) with $t_0=T$ shows that $\wp_n(\omega_2)-\wp_n(\omega_1)\le 2T\delta_{\ell-1}^{-1}\|\omega_1-\omega_2\|_{0,T}\le\kappa_n\|\omega_1-\omega_2\|_{0,T}$. On the other hand, suppose that the set $\big\{t\in[a_n,T):t\ge a_n+\kappa_n\|\omega_1-\omega_2\|_{0,t}\big\}$ is not empty and contains t_0. since $\vartheta_{\ell-1}(\omega_1)=\wp_n(\omega_1)-j2^{1-\ell}T\le\wp_n(\omega_2)-j2^{1-\ell}T\le\vartheta_{\ell-1}(\omega_2)$, we see that $\mathfrak{a}_{\ell-1}=\vartheta_{\ell-1}(\omega_1)$ and can deduce that

$$t_0\ge a_n+\kappa_n\|\omega_1-\omega_2\|_{0,t_0}=\vartheta_{\ell-1}(\omega_1)+j2^{1-\ell}T+\kappa_n\|\omega_1-\omega_2\|_{0,t_0}>\vartheta_{\ell-1}(\omega_1)+2T\delta_{\ell-1}^{-1}\|\omega_1-\omega_2\|_{0,t_0}=\mathfrak{a}_{\ell-1}+2T\delta_{\ell-1}^{-1}\|\omega_1-\omega_2\|_{0,t_0}.$$

Then (6.50) and (6.48) imply that $\wp_n(\omega_2)-\wp_n(\omega_1)\le 2T\delta_{\ell-1}^{-1}\|\omega_1-\omega_2\|_{0,t_0}\le\kappa_n\|\omega_1-\omega_2\|_{0,t_0}$.
ii) When $\wp_n(\omega_1)=\vartheta_\ell(\omega_1)$, applying (6.48) with $t_0=T$ shows that $\wp_n(\omega_2)-\wp_n(\omega_1)\le\vartheta_\ell(\omega_2)-\vartheta_\ell(\omega_1)\le 2T\delta_\ell^{-1}\|\omega_1-\omega_2\|_{0,T}=\kappa_n\|\omega_1-\omega_2\|_{0,T}$. Next, suppose that the set $\big\{t\in[a_n,T):t\ge a_n+\kappa_n\|\omega_1-\omega_2\|_{0,t}\big\}$ is not empty and contains t_0. Since $\vartheta_\ell(\omega_1)=\wp_n(\omega_1)\le\wp_n(\omega_2)\le\vartheta_\ell(\omega_2)$. we see that $\mathfrak{a}_\ell=\vartheta_\ell(\omega_1)$ and can deduce that $t_0\ge a_n+\kappa_n\|\omega_1-\omega_2\|_{0,t_0}=\vartheta_\ell(\omega_1)+\kappa_n\|\omega_1-\omega_2\|_{0,t_0}=\mathfrak{a}_\ell+2T\delta_\ell^{-1}\|\omega_1-\omega_2\|_{0,t_0}$. Applying (6.48) again yields that $\wp_n(\omega_2)-\wp_n(\omega_1)\le\vartheta_\ell(\omega_2)-\vartheta_\ell(\omega_1)\le 2T\delta_\ell^{-1}\|\omega_1-\omega_2\|_{0,t_0}=\kappa_n\|\omega_1-\omega_2\|_{0,t_0}$. □

Proof of Lemma 5.1: Fix $n,k\in\mathbb{N}$. We define $H_t:=1\wedge(2^k(t-\wp_n)-1)^+$ and $\Delta_t:=U_t-L_t$, $t\in[0,T]$. Let $(t_1,\omega_1),(t_2,\omega_2)\in[0,T]\times\Omega$. We set $\mathbf{d}^{1,2}:=\mathbf{d}_\infty\big((t_1,\omega_1),(t_2,\omega_2)\big)$ and assume without loss of generality that $t_1\le t_2$.

Since (2.5) shows that $\big|H_{t_1}(\omega_2)-H_{t_2}(\omega_2)\big|\le\big|\big(2^k(t_1-\wp_n(\omega_2))-1\big)^+-\big(2^k(t_2-\wp_n(\omega_2))-1\big)^+\big|\le 2^k|t_1-t_2|$, (2.2) implies that

$$\begin{aligned}\big|Y^{n,k}_{t_1}(\omega_2)-Y^{n,k}_{t_2}(\omega_2)\big|&\le\big|L_{t_1}(\omega_2)-L_{t_2}(\omega_2)\big|+\big|H_{t_1}(\omega_2)-H_{t_2}(\omega_2)\big||\Delta_{t_1}(\omega_2)|+H_{t_2}(\omega_2)|\Delta_{t_1}(\omega_2)-\Delta_{t_2}(\omega_2)|\\&\le\rho_0\big(\mathbf{d}_\infty\big((t_1,\omega_2),(t_2,\omega_2)\big)\big)+2^{1+k}M_0|t_1-t_2|+2\rho_0\big(\mathbf{d}_\infty\big((t_1,\omega_2),(t_2,\omega_2)\big)\big).\end{aligned}\tag{6.51}$$

Since

$$\begin{aligned}\sup_{r\in[t_1,t_2]}\big|\omega_2(r)-\omega_2(t_1)\big|&\le|\omega_1(t_1)-\omega_2(t_1)|+\sup_{r\in[t_1,t_2]}|\omega_2(r)-\omega_1(t_1)|\le 2\Big(\|\omega_1-\omega_2\|_{0,t_1}\vee\sup_{r\in[t_1,t_2]}|\omega_1(t_1)-\omega_2(r)|\Big)\\&=2\|\omega_1(\cdot\wedge t_1)-\omega_2(\cdot\wedge t_2)\|_{0,T},\end{aligned}$$

one can deduce that $\mathbf{d}_\infty\big((t_1,\omega_2),(t_2,\omega_2)\big)=|t_1-t_2|+\sup_{r\in[t_1,t_2]}\big|\omega_2(r)-\omega_2(t_1)\big|\le 2\big(|t_1-t_2|+\|\omega_1(\cdot\wedge t_1)-\omega_2(\cdot\wedge t_2)\|_{0,T}\big)=2\mathbf{d}^{1,2}$. Then it follows from (6.51) that

$$\big|Y^{n,k}_{t_1}(\omega_2)-Y^{n,k}_{t_2}(\omega_2)\big|\le 3\rho_0\big(2\mathbf{d}^{1,2}\big)+2^{1+k}M_0\mathbf{d}^{1,2}.\tag{6.52}$$

Since (2.5), Proposition 5.1 (2) imply that

$$\big|H_{t_1}(\omega_1)-H_{t_1}(\omega_2)\big|\le 2^k\kappa_n\|\omega_1-\omega_2\|_{0,t_1}, \tag{6.53*}$$

and since $\|\omega_1-\omega_2\|_{0,t_1}\le\|\omega_1(\cdot\wedge t_1)-\omega_2(\cdot\wedge t_2)\|_{0,T}\le\mathbf{d}^{1,2}$, we can further deduce that

$$\begin{aligned}\big|Y^{n,k}_{t_1}(\omega_1)-Y^{n,k}_{t_1}(\omega_2)\big| &\le \big|L_{t_1}(\omega_1)-L_{t_1}(\omega_2)\big|+\big|H_{t_1}(\omega_1)-H_{t_1}(\omega_2)\big||\Delta_{t_1}(\omega_1)|+H_{t_1}(\omega_2)|\Delta_{t_1}(\omega_1)-\Delta_{t_1}(\omega_2)|\\ &\le \rho_0\big(\|\omega_1-\omega_2\|_{0,t_1}\big)+2^{1+k}M_0\kappa_n\|\omega_1-\omega_2\|_{0,t_1}+2\rho_0\big(\|\omega_1-\omega_2\|_{0,t_1}\big)\le 3\rho_0(2\mathbf{d}^{1,2})+2^{1+k}M_0\kappa_n\mathbf{d}^{1,2},\end{aligned}$$

which together with (6.52) leads to that $\big|Y^{n,k}_{t_1}(\omega_1)-Y^{n,k}_{t_2}(\omega_2)\big|\le 6\rho_0(2\mathbf{d}^{1,2})+2^{1+k}M_0(1+\kappa_n)\mathbf{d}^{1,2}=\rho_{n,k}(\mathbf{d}^{1,2})$. □

Proof of (5.4)**:** Fix $(t,\omega)\in[0,T]\times\Omega$. We will simply denote 2^{1-k} by δ and denote the term $U\big((\wp_n(\omega)+\delta)\wedge t,\omega\big)-U(\wp_n(\omega)\wedge t,\omega)$ by Δ^U. Let $(\mathbb{P},\gamma,\nu)\in\mathcal{P}_t\times\mathcal{T}^t\times\mathcal{T}^t$ and define

$$J_{\gamma,\nu}(\widetilde\omega):=\mathbf{1}_{\{\gamma(\widetilde\omega)>\wp_n(\omega\otimes_t\widetilde\omega)\}}\Big(U\big((\wp_n(\omega\otimes_t\widetilde\omega)+\delta)\wedge(\nu(\widetilde\omega)\vee\wp_n(\omega\otimes_t\widetilde\omega)),\omega\otimes_t\widetilde\omega\big)-U\big(\wp_n(\omega\otimes_t\widetilde\omega),\omega\otimes_t\widetilde\omega\big)\Big),\quad\forall\,\widetilde\omega\in\Omega^t.$$

1) *We first show by three cases that*

$$\mathbb{E}_{\mathbb{P}}\big[|J_{\gamma,\nu}-\Delta^U|\big]\le\widehat\rho_0(\delta). \tag{6.54}$$

(i) When $\wp_n(\omega)<t-\delta$, applying Lemma A.1 with $(t,s,\tau)=(0,t,\wp_n)$ yields that $t_n:=\wp_n(\omega)=\wp_n(\omega\otimes_t\widetilde\omega)$, $\forall\,\widetilde\omega\in\Omega^t$. Since U is an $\mathbf{F}-$adapted process by (A1) and (2.3), one has $U_{t_n}\in\mathcal{F}_{t_n}\subset\mathcal{F}_t$ and $U_{t_n+\delta}\in\mathcal{F}_{t_n+\delta}\subset\mathcal{F}_t$. Let $\widetilde\omega\in\Omega^t$. Using (2.6) with $(t,s,\eta)=(0,t,U_{t_n})$ and $(t,s,\eta)=(0,t,U_{t_n+\delta})$ respectively shows that $U(t_n,\omega\otimes_t\widetilde\omega)=U(t_n,\omega)$ and $U\big(t_n+\delta,\omega\otimes_t\widetilde\omega\big)=U\big(t_n+\delta,\omega\big)$. As $t_n+\delta<t\le\gamma(\widetilde\omega)\wedge\nu(\widetilde\omega)$, one has

$$J_{\gamma,\nu}(\widetilde\omega)=\mathbf{1}_{\{\gamma(\widetilde\omega)>t_n\}}\big(U\big((t_n+\delta)\wedge(\nu(\widetilde\omega)\vee t_n),\omega\otimes_t\widetilde\omega\big)-U(t_n,\omega\otimes_t\widetilde\omega)\big)=U\big(t_n+\delta,\omega\big)-U(t_n,\omega)=\Delta^U.$$

(ii) When $t-\delta\le\wp_n(\omega)<t$, we still have $t_n=\wp_n(\omega)=\wp_n(\omega\otimes_t\widetilde\omega)$ and $U(t_n,\omega\otimes_t\widetilde\omega)=U(t_n,\omega)$, $\forall\,\widetilde\omega\in\Omega^t$. Set $\nu_n:=(t_n+\delta)\wedge\nu\in\mathcal{T}^t$. For any $\widetilde\omega\in\Omega^t$, we see from $t_n<t\le\gamma(\widetilde\omega)\wedge\nu(\widetilde\omega)$ that

$$J_{\gamma,\nu}(\widetilde\omega)-\Delta^U=\mathbf{1}_{\{\gamma(\widetilde\omega)>t_n\}}\Big(U\big((t_n+\delta)\wedge(\nu(\widetilde\omega)\vee t_n),\omega\otimes_t\widetilde\omega\big)-U(t_n,\omega\otimes_t\widetilde\omega)\Big)-U(t,\omega)+U(t_n,\omega)=U\big(\nu_n(\widetilde\omega),\omega\otimes_t\widetilde\omega\big)-U(t,\omega).$$

Since $t\le\nu_n(\widetilde\omega)\le(t_n+\delta)\wedge T\le(t+\delta)\wedge T$, one can further deduce from (2.2) that

$$\begin{aligned}\big|J_{\gamma,\nu}(\widetilde\omega)-\Delta^U\big|&\le\rho_0\Big((\nu_n(\widetilde\omega)-t)+\sup_{r\in[0,T]}\big|(\omega\otimes_t\widetilde\omega)\big(r\wedge\nu_n(\widetilde\omega)\big)-\omega(r\wedge t)\big|\Big)\le\rho_0\Big(\delta+\sup_{r\in[t,(t+\delta)\wedge T]}|\widetilde\omega(r)|\Big)\\ &=\rho_0\Big(\delta+\sup_{r\in[t,(t+\delta)\wedge T]}\big|B^t_r(\widetilde\omega)-B^t_t(\widetilde\omega)\big|\Big).\end{aligned}$$

Taking expectation $\mathbb{E}_{\mathbb{P}}[\;]$, we see from (3.5) that $\mathbb{E}_{\mathbb{P}}\big[|J_{\gamma,\nu}-\Delta^U|\big]\le\widehat\rho_0(\delta)$.
(iii) When $\wp_n(\omega)\ge t$, we see that $\Delta^U=U(t,\omega)-U(t,\omega)=0$. As Lemma A.1 shows that $\wp^{t,\omega}_n\in\mathcal{T}^t$, $\zeta_n:=(\wp^{t,\omega}_n+\delta)\wedge(\nu\vee\wp^{t,\omega}_n)$ is also an $\mathbf{F}^t-$stopping time. Given $\widetilde\omega\in\Omega^t$, we set $s^1_n:=\wp^{t,\omega}_n(\widetilde\omega)\le\zeta_n(\widetilde\omega):=s^2_n$. Since $s^2_n\le\wp^{t,\omega}_n(\widetilde\omega)+\delta=s^1_n+\delta$, applying (2.2) again yields that

$$\begin{aligned}\big|J_{\gamma,\nu}(\widetilde\omega)-\Delta^U\big|=\big|J_{\gamma,\nu}(\widetilde\omega)\big|&=\mathbf{1}_{\{\gamma(\widetilde\omega)>\wp^{t,\omega}_n(\widetilde\omega)\}}\Big|U\big((\wp^{t,\omega}_n(\widetilde\omega)+\delta)\wedge(\nu(\widetilde\omega)\vee\wp^{t,\omega}_n(\widetilde\omega)),\omega\otimes_t\widetilde\omega\big)-U\big(\wp^{t,\omega}_n(\widetilde\omega),\omega\otimes_t\widetilde\omega\big)\Big|\\ &\le\big|U\big(s^2_n,\omega\otimes_t\widetilde\omega\big)-U\big(s^1_n,\omega\otimes_t\widetilde\omega\big)\big|\le\rho_0\Big((s^2_n-s^1_n)+\sup_{r\in[0,T]}\big|(\omega\otimes_t\widetilde\omega)(r\wedge s^2_n)-(\omega\otimes_t\widetilde\omega)(r\wedge s^1_n)\big|\Big)\\ &=\rho_0\Big((s^2_n-s^1_n)+\sup_{r\in[s^1_n,s^2_n]}\big|\widetilde\omega(r)-\widetilde\omega(s^1_n)\big|\Big)\le\rho_0\Big(\delta+\sup_{r\in[\wp^{t,\omega}_n(\widetilde\omega),(\wp^{t,\omega}_n(\widetilde\omega)+\delta)\wedge T]}\big|B^t_r(\widetilde\omega)-B^t\big(\wp^{t,\omega}_n(\widetilde\omega),\widetilde\omega\big)\big|\Big).\end{aligned}$$

Taking expectation $\mathbb{E}_{\mathbb{P}}[\;]$ and using (3.5) yield that $\mathbb{E}_{\mathbb{P}}\big[|J_{\gamma,\nu}-\Delta^U|\big]\le\widehat\rho_0(\delta)$. Hence, we proved (6.54).
2) *Next, we use* (6.54) *to verify* (5.4).
2a) For any $(t',\omega')\in[0,T]\times\Omega$, since (5.2) and (A2) imply that $L(t',\omega')\le Y^{n,k}(t',\omega')\le U(t',\omega')$,

$$\begin{aligned}\widehat Y^{n,k}(t',\omega')&=\mathbf{1}_{\{t'\le\wp_n(\omega')\}}L(t',\omega')+\mathbf{1}_{\{t'>\wp_n(\omega')\}}Y^{n,k}\big(\wp^{n,k}(\omega')\wedge t',\omega'\big)\\ &\le\mathbf{1}_{\{t'\le\wp_n(\omega')\}}L(t',\omega')+\mathbf{1}_{\{t'>\wp_n(\omega')\}}U\big((\wp_n(\omega')+\delta)\wedge t',\omega'\big).\end{aligned} \tag{6.55}$$

Given $(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t$ and $\widetilde{\omega}\in\Omega^t$, taking $(t',\omega')=\big(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)$ in (6.55) yields that

$$\begin{aligned}&\big(\widehat{Y}^{n,k}\big)^{t,\omega}_{\gamma}(\widetilde{\omega})-(\mathscr{Y}^n)^{t,\omega}_{\gamma}(\widetilde{\omega})=\widehat{Y}^{n,k}\big(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)-\mathscr{Y}^n\big(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)\\&\quad\le\mathbf{1}_{\{\gamma(\widetilde{\omega})>\wp_n(\omega\otimes_t\widetilde{\omega})\}}\Big(U\big((\wp_n(\omega\otimes_t\widetilde{\omega})+\delta)\wedge(\gamma(\widetilde{\omega})\vee\wp_n(\omega\otimes_t\widetilde{\omega})),\omega\otimes_t\widetilde{\omega}\big)-U\big(\wp_n(\omega\otimes_t\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)\Big)=J_{\gamma,\gamma}(\widetilde{\omega}).\end{aligned}$$

It then follows from (6.54) that $\mathbb{E}_\mathbb{P}\Big[\big(\widehat{Y}^{n,k}\big)^{t,\omega}_{\gamma}\Big]\le\mathbb{E}_\mathbb{P}\big[(\mathscr{Y}^n)^{t,\omega}_{\gamma}+J_{\gamma,\gamma}\big]\le\mathscr{Z}^n_t(\omega)+\Delta^U+\widehat{\rho}_0(\delta)$. Taking supremum over $(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t$ on the left-hand-side leads to that $Z^{n,k}_t(\omega)\le\mathscr{Z}^n_t(\omega)+\Delta^U+\widehat{\rho}_0(\delta)$.

2b) To show the left-hand-side of (5.4), we let $(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t$ and set $\widetilde{\gamma}:=\big(\gamma+\delta\big)\wedge T\in\mathcal{T}^t$. Also, let $(t',\omega')\in[0,T]\times\Omega$, one has

$$\mathscr{Y}^n(t',\omega')\le\mathbf{1}_{\{t'\le\wp_n(\omega')\}}U(t',\omega')+\mathbf{1}_{\{t'>\wp_n(\omega')\}}U\big(\wp_n(\omega'),\omega'\big)=U\big(\wp_n(\omega')\wedge t',\omega'\big).\tag{6.56}$$

If $t'\le T-\delta$, since

$$\begin{aligned}\widehat{Y}^{n,k}(t'+\delta,\omega')&=\mathbf{1}_{\{t'\le\wp_n(\omega')-2^{-k}\}}L(t'+\delta,\omega')+\mathbf{1}_{\{t'\ge\wp_n(\omega')\}}U\big((\wp_n(\omega')+\delta)\wedge T,\omega'\big)\\&\quad+\mathbf{1}_{\{\wp_n(\omega')-2^{-k}<t'<\wp_n(\omega')\}}\Big\{\big[1-2^k(t'+2^{-k}-\wp_n(\omega'))\big]L(t'+\delta,\omega')+2^k(t'+2^{-k}-\wp_n(\omega'))U(t'+\delta,\omega')\Big\}\\&\ge\mathbf{1}_{\{t'\le\wp_n(\omega')\}}L(t'+\delta,\omega')+\mathbf{1}_{\{t'>\wp_n(\omega')\}}U\big((\wp_n(\omega')+\delta)\wedge T,\omega'\big),\end{aligned}$$

we can obtain that

$$\begin{aligned}\mathscr{Y}^n(t',\omega')-\widehat{Y}^{n,k}(t'+\delta,\omega')&\le\mathbf{1}_{\{t'\le\wp_n(\omega')\}}\big(L(t',\omega')-L(t'+\delta,\omega')\big)\\&\quad+\mathbf{1}_{\{t'>\wp_n(\omega')\}}\big(U(\wp_n(\omega'),\omega')-U\big((\wp_n(\omega')+\delta)\wedge T,\omega'\big)\big).\end{aligned}\tag{6.57}$$

Also, (5.3) and (A2) imply that

$$\widehat{Y}^{n,k}(T,\omega')=\mathbf{1}_{\{\wp_n(\omega')>T-\delta\}}U(T,\omega')+\mathbf{1}_{\{\wp_n(\omega')\le T-\delta\}}U\big((\wp_n(\omega')+\delta)\wedge T,\omega'\big)=U\big((\wp_n(\omega')+\delta)\wedge T,\omega'\big).\tag{6.58}$$

Let $\widetilde{\omega}\in\{\gamma>T-\delta\}$, so $\widetilde{\gamma}(\widetilde{\omega})=T$. Taking $(t',\omega')=\big(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)$ in (6.56), (6.58) and using (2.2) yield that

$$\begin{aligned}&(\mathscr{Y}^n)^{t,\omega}_{\gamma}(\widetilde{\omega})-\big(\widehat{Y}^{n,k}\big)^{t,\omega}_{\widetilde{\gamma}}(\widetilde{\omega})=\mathscr{Y}^n\big(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)-\widehat{Y}^{n,k}(T,\omega\otimes_t\widetilde{\omega})\le U\big(\wp_n(\omega\otimes_t\widetilde{\omega})\wedge\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)-U\big((\wp_n(\omega\otimes_t\widetilde{\omega})+\delta)\wedge T,\omega\otimes_t\widetilde{\omega}\big)\\&=\mathbf{1}_{\{\gamma(\widetilde{\omega})\le\wp_n(\omega\otimes_t\widetilde{\omega})\}}\Big(U\big(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)-U\big(T,\omega\otimes_t\widetilde{\omega}\big)\Big)+\mathbf{1}_{\{\gamma(\widetilde{\omega})>\wp_n(\omega\otimes_t\widetilde{\omega})\}}\Big(U\big(\wp_n(\omega\otimes_t\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)-U\big((\wp_n(\omega\otimes_t\widetilde{\omega})+\delta)\wedge T,\omega\otimes_t\widetilde{\omega}\big)\Big)\\&\le\rho_0\Big(\big(T-\gamma(\widetilde{\omega})\big)+\sup_{r\in[\gamma(\widetilde{\omega}),T]}\big|\widetilde{\omega}(r)-\widetilde{\omega}(\gamma(\widetilde{\omega}))\big|\Big)-J_{\gamma,T}(\widetilde{\omega})\le\rho_0\Big(\delta+\sup_{r\in[\gamma(\widetilde{\omega}),(\gamma(\widetilde{\omega})+\delta)\wedge T]}\big|B^t_r(\widetilde{\omega})-B^t_\gamma(\widetilde{\omega})\big|\Big)-J_{\gamma,T}(\widetilde{\omega}).\end{aligned}\tag{6.59}$$

On the other hand, let $\widetilde{\omega}\in\{\gamma\le T-\delta\}$. applying (6.57) with $(t',\omega')=\big(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)$ and using (2.2) yield that

$$\begin{aligned}&(\mathscr{Y}^n)^{t,\omega}_{\gamma}(\widetilde{\omega})-\big(\widehat{Y}^{n,k}\big)^{t,\omega}_{\widetilde{\gamma}}(\widetilde{\omega})=\mathscr{Y}^n\big(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)-\widehat{Y}^{n,k}\big(\gamma(\widetilde{\omega})+\delta,\omega\otimes_t\widetilde{\omega}\big)\\&\quad\le\mathbf{1}_{\{\gamma(\widetilde{\omega})\le\wp_n(\omega\otimes_t\widetilde{\omega})\}}\Big(L(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega})-L(\gamma(\widetilde{\omega})+\delta,\omega\otimes_t\widetilde{\omega})\Big)\\&\qquad+\mathbf{1}_{\{\gamma(\widetilde{\omega})>\wp_n(\omega\otimes_t\widetilde{\omega})\}}\Big(U(\wp_n(\omega\otimes_t\widetilde{\omega}),\omega\otimes_t\widetilde{\omega})-U\big((\wp_n(\omega\otimes_t\widetilde{\omega})+\delta)\wedge T,\omega\otimes_t\widetilde{\omega}\big)\Big)\\&\quad\le\rho_0\Big(\delta+\sup_{r\in[\gamma(\widetilde{\omega}),(\gamma(\widetilde{\omega})+\delta)\wedge T]}\big|\widetilde{\omega}(r)-\widetilde{\omega}(\gamma(\widetilde{\omega}))\big|\Big)-J_{\gamma,T}(\widetilde{\omega})=\rho_0\Big(\delta+\sup_{r\in[\gamma(\widetilde{\omega}),(\gamma(\widetilde{\omega})+\delta)\wedge T]}\big|B^t_r(\widetilde{\omega})-B^t_\gamma(\widetilde{\omega})\big|\Big)-J_{\gamma,T}(\widetilde{\omega}).\end{aligned}$$

Combining this with (6.59), we see from (6.54) and (3.5) that

$$\mathbb{E}_\mathbb{P}\big[(\mathscr{Y}^n)^{t,\omega}_{\gamma}\big]\le\mathbb{E}_\mathbb{P}\Big[\big(\widehat{Y}^{n,k}\big)^{t,\omega}_{\widetilde{\gamma}}-J_{\gamma,T}\Big]+\widehat{\rho}_0(\delta)\le Z^{n,k}_t(\omega)-\Delta^U+2\widehat{\rho}_0(\delta).$$

Then taking supremum over $(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t$ on the left-hand-side leads to that $\mathscr{Z}^n_t(\omega)\le Z^{n,k}_t(\omega)-\Delta^U+2\widehat{\rho}_0(\delta)$. $\square$

Proof of (5.5): Fix $(t,\omega)\in[0,T]\times\Omega$. We will simply denote $\frac{2T}{n+3}$ by δ and denote the term $U\big(\wp_{n+1}(\omega)\wedge t,\omega\big)-U\big(\wp_n(\omega)\wedge t,\omega\big)$ by $\widetilde{\Delta}^U$. Let $(\mathbb{P},\gamma,\nu)\in\mathcal{P}_t\times\mathcal{T}^t\times\mathcal{T}^t$ and define

$$J_{\gamma,\nu}(\widetilde{\omega}):=\mathbf{1}_{\{\gamma(\widetilde{\omega})>\wp_n(\omega\otimes_t\widetilde{\omega})\}}\Big(U\big(\wp_{n+1}(\omega\otimes_t\widetilde{\omega})\wedge(\nu(\widetilde{\omega})\vee\wp_n(\omega\otimes_t\widetilde{\omega})),\omega\otimes_t\widetilde{\omega}\big)-U\big(\wp_n(\omega\otimes_t\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)\Big),\quad\forall\,\widetilde{\omega}\in\Omega^t.$$

In light of Proposition 5.1 (1), one can deduce (6.54) again by three cases: $\wp_{n+1}(\omega)<t$, $\wp_n(\omega)<t\le\wp_{n+1}(\omega)$ and $\wp_n(\omega)\ge t$.

1) *Let us show the right-hand-side of* (5.5) *first.*

For any $(t',\omega')\in[0,T]\times\Omega$, since an analogy to (6.56) shows that $\mathscr{Y}^{n+1}(t',\omega')\le U\big(\wp_{n+1}(\omega')\wedge t',\omega'\big)$, we have

$$\begin{aligned}\mathscr{Y}^{n+1}(t',\omega')&=\mathbf{1}_{\{t'\le\wp_n(\omega')\}}L(t',\omega')+\mathbf{1}_{\{t'>\wp_n(\omega')\}}\mathscr{Y}^{n+1}\big(t',\omega'\big)\\&\le\mathbf{1}_{\{t'\le\wp_n(\omega')\}}L(t',\omega')+\mathbf{1}_{\{t'>\wp_n(\omega')\}}U\big(\wp_{n+1}(\omega')\wedge t',\omega'\big).\end{aligned}\tag{6.60}$$

Given $(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t$ and $\widetilde{\omega}\in\Omega^t$, taking $(t',\omega')=\big(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)$ in (6.60) yields that

$$\begin{aligned}&\big(\mathscr{Y}^{n+1}\big)^{t,\omega}_\gamma(\widetilde{\omega})-(\mathscr{Y}^n)^{t,\omega}_\gamma(\widetilde{\omega})=\mathscr{Y}^{n+1}\big(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)-\mathscr{Y}^n\big(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)\\&\quad\le\mathbf{1}_{\{\gamma(\widetilde{\omega})>\wp_n(\omega\otimes_t\widetilde{\omega})\}}\Big(U\big(\wp_{n+1}(\omega\otimes_t\widetilde{\omega})\wedge(\gamma(\widetilde{\omega})\vee\wp_n(\omega\otimes_t\widetilde{\omega})),\omega\otimes_t\widetilde{\omega}\big)-U\big(\wp_n(\omega\otimes_t\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)\Big)=J_{\gamma,\gamma}(\widetilde{\omega}).\end{aligned}$$

It then follows from (6.54) that $\mathbb{E}_\mathbb{P}\big[\big(\mathscr{Y}^{n+1}\big)^{t,\omega}_\gamma\big]\le\mathbb{E}_\mathbb{P}\big[(\mathscr{Y}^n)^{t,\omega}_\gamma+J_{\gamma,\gamma}\big]\le\mathscr{Z}^n_t(\omega)+\widetilde{\Delta}^U+\widehat{\rho}_0(\delta)$. Taking supremum over $(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t$ on the left-hand-side leads to that $\mathscr{Z}^{n+1}_t(\omega)\le\mathscr{Z}^n_t(\omega)+\widetilde{\Delta}^U+\widehat{\rho}_0(\delta)$.

2) To show the left hand side of (5.5), we let $(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t$ and set $\widetilde{\gamma}:=\big(\gamma+\delta\big)\wedge T\in\mathcal{T}^t$. We also let $(t',\omega')\in[0,T]\times\Omega$. If $t'\le T-\delta$, since $\wp_{n+1}(\omega')\le\wp_n(\omega')+\delta$ by Proposition 5.1 (1), one can deduce that

$$\begin{aligned}\mathscr{Y}^{n+1}(t'+\delta,\omega')&=\mathbf{1}_{\{t'+\delta\le\wp_{n+1}(\omega')\}}L(t'+\delta,\omega')+\mathbf{1}_{\{t'+\delta>\wp_{n+1}(\omega')\}}U\big(\wp_{n+1}(\omega'),\omega'\big)\\&\ge\mathbf{1}_{\{t'+\delta\le\wp_{n+1}(\omega')\}}L(t'+\delta,\omega')+\mathbf{1}_{\{\wp_{n+1}(\omega')-\delta<t'\le\wp_n(\omega')\}}L\big(\wp_{n+1}(\omega'),\omega'\big)+\mathbf{1}_{\{t'>\wp_n(\omega')\}}U\big(\wp_{n+1}(\omega'),\omega'\big),\end{aligned}$$

and thus that

$$\begin{aligned}\mathscr{Y}^n(t',\omega')-\mathscr{Y}^{n+1}(t'+\delta,\omega')&\le\mathbf{1}_{\{t'+\delta\le\wp_{n+1}(\omega')\}}\big(L(t',\omega')-L(t'+\delta,\omega')\big)+\mathbf{1}_{\{\wp_{n+1}(\omega')-\delta<t'\le\wp_n(\omega')\}}\Big(L(t',\omega')-L\big(\wp_{n+1}(\omega')\vee t',\omega'\big)\Big)\\&\quad+\mathbf{1}_{\{t'>\wp_n(\omega')\}}\Big(U\big(\wp_n(\omega'),\omega'\big)-U\big(\wp_{n+1}(\omega'),\omega'\big)\Big).\end{aligned}\tag{6.61}$$

Also, (A2) implies that

$$\begin{aligned}\mathscr{Y}^{n+1}(T,\omega')&=\mathbf{1}_{\{T=\wp_{n+1}(\omega')\}}L(T,\omega')+\mathbf{1}_{\{T>\wp_{n+1}(\omega')\}}U\big(\wp_{n+1}(\omega'),\omega'\big)\\&=\mathbf{1}_{\{T=\wp_{n+1}(\omega')\}}U(T,\omega')+\mathbf{1}_{\{T>\wp_{n+1}(\omega')\}}U\big(\wp_{n+1}(\omega'),\omega'\big)=U\big(\wp_{n+1}(\omega'),\omega'\big).\end{aligned}\tag{6.62}$$

Let $\widetilde{\omega}\in\{\gamma>T-\delta\}$, so $\widetilde{\gamma}(\widetilde{\omega})=T$. Taking $(t',\omega')=\big(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)$ in (6.56) and (6.62) yields that

$$\begin{aligned}(\mathscr{Y}^n)^{t,\omega}_\gamma(\widetilde{\omega})-\big(\mathscr{Y}^{n+1}\big)^{t,\omega}_{\widetilde{\gamma}}(\widetilde{\omega})&=\mathscr{Y}^n\big(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)-\mathscr{Y}^{n+1}(T,\omega\otimes_t\widetilde{\omega})\le U\big(\wp_n(\omega\otimes_t\widetilde{\omega})\wedge\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)-U\big(\wp_{n+1}(\omega\otimes_t\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)\\&=\mathbf{1}_{\{\gamma(\widetilde{\omega})\le\wp_n(\omega\otimes_t\widetilde{\omega})\}}\Big(U\big(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)-U\big(\wp_{n+1}(\omega\otimes_t\widetilde{\omega})\vee\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)\Big)\\&\quad+\mathbf{1}_{\{\gamma(\widetilde{\omega})>\wp_n(\omega\otimes_t\widetilde{\omega})\}}\Big(U\big(\wp_n(\omega\otimes_t\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)-U\big(\wp_{n+1}(\omega\otimes_t\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)\Big)\\&\le\rho_0\Big(\delta+\sup_{r\in[\gamma(\widetilde{\omega}),(\gamma(\widetilde{\omega})+\delta)\wedge T]}\big|B^t_r(\widetilde{\omega})-B^t_\gamma(\widetilde{\omega})\big|\Big)-J_{\gamma,T}(\widetilde{\omega}),\end{aligned}\tag{6.63}$$

where we obtained from (2.2) that

$$\begin{aligned}&U\big(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)-U\big(\wp_{n+1}(\omega\otimes_t\widetilde{\omega})\vee\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)\\&\quad\le\rho_0\Big(\big(\wp_{n+1}(\omega\otimes_t\widetilde{\omega})\vee\gamma(\widetilde{\omega})-\gamma(\widetilde{\omega})\big)+\sup_{r\in[0,T]}\big|(\omega\otimes_t\widetilde{\omega})\big(r\wedge(\wp_{n+1}(\omega\otimes_t\widetilde{\omega})\vee\gamma(\widetilde{\omega}))\big)-(\omega\otimes_t\widetilde{\omega})(r\wedge\gamma(\widetilde{\omega}))\big|\Big)\\&\quad\le\rho_0\Big(\big(T-\gamma(\widetilde{\omega})\big)+\sup_{r\in[\gamma(\widetilde{\omega}),T]}\big|\widetilde{\omega}(r)-\widetilde{\omega}(\gamma(\widetilde{\omega}))\big|\Big)\le\rho_0\Big(\delta+\sup_{r\in[\gamma(\widetilde{\omega}),(\gamma(\widetilde{\omega})+\delta)\wedge T]}\big|B^t_r(\widetilde{\omega})-B^t_\gamma(\widetilde{\omega})\big|\Big).\end{aligned}$$

On the other hand, let $\widetilde{\omega}\in\{\gamma\le T-\delta\}$. applying (6.61) with $(t',\omega')=\big(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)$ yields that

$$\begin{aligned}(\mathscr{Y}^n)^{t,\omega}_{\gamma}(\widetilde{\omega})-\big(\mathscr{Y}^{n+1}\big)^{t,\omega}_{\widetilde{\gamma}}(\widetilde{\omega})&=\mathscr{Y}^n\big(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)-\mathscr{Y}^{n+1}\big(\gamma(\widetilde{\omega})+\delta,\omega\otimes_t\widetilde{\omega}\big)\\&\le \mathbf{1}_{\{\gamma(\widetilde{\omega})+\delta\le\wp_{n+1}(\omega\otimes_t\widetilde{\omega})\}}\Big(L(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega})-L(\gamma(\widetilde{\omega})+\delta,\omega\otimes_t\widetilde{\omega})\Big)\\&\quad+\mathbf{1}_{\{\wp_{n+1}(\omega\otimes_t\widetilde{\omega})-\delta<\gamma(\widetilde{\omega})\le\wp_n(\omega\otimes_t\widetilde{\omega})\}}\Big(L(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega})-L\big(\wp_{n+1}(\omega\otimes_t\widetilde{\omega})\vee\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)\Big)\\&\quad+\mathbf{1}_{\{\gamma(\widetilde{\omega})>\wp_n(\omega\otimes_t\widetilde{\omega})\}}\Big(U(\wp_n(\omega\otimes_t\widetilde{\omega}),\omega\otimes_t\widetilde{\omega})-U\big(\wp_{n+1}(\omega\otimes_t\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)\Big)\\&\le\rho_0\Big(\delta+\sup_{r\in[\gamma(\widetilde{\omega}),(\gamma(\widetilde{\omega})+\delta)\wedge T]}\big|B^t_r(\widetilde{\omega})-B^t_{\gamma}(\widetilde{\omega})\big|\Big)-J_{\gamma,T}(\widetilde{\omega}),\end{aligned}\tag{6.64}$$

where we derived from (2.2) that if $\wp_{n+1}(\omega\otimes\widetilde{\omega})<\gamma(\widetilde{\omega})+\delta$,

$$\begin{aligned}&L\big(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)-L\big(\wp_{n+1}(\omega\otimes_t\widetilde{\omega})\vee\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)\\&\quad\le\ \rho_0\Big(\big(\wp_{n+1}(\omega\otimes_t\widetilde{\omega})\vee\gamma(\widetilde{\omega})-\gamma(\widetilde{\omega})\big)+\sup_{r\in[0,T]}\big|(\omega\otimes_t\widetilde{\omega})\big(r\wedge(\wp_{n+1}(\omega\otimes_t\widetilde{\omega})\vee\gamma(\widetilde{\omega}))\big)-(\omega\otimes_t\widetilde{\omega})(r\wedge\gamma(\widetilde{\omega}))\big|\Big)\\&\quad\le\ \rho_0\Big(\delta+\sup_{r\in[\gamma(\widetilde{\omega}),(\gamma(\widetilde{\omega})+\delta)\wedge T]}\big|\widetilde{\omega}(r)-\widetilde{\omega}(\gamma(\widetilde{\omega}))\big|\Big)\le\rho_0\Big(\delta+\sup_{r\in[\gamma(\widetilde{\omega}),(\gamma(\widetilde{\omega})+\delta)\wedge T]}\big|B^t_r(\widetilde{\omega})-B^t_{\gamma}(\widetilde{\omega})\big|\Big).\end{aligned}$$

Combining (6.63) with (6.64), we see from (6.54) and (3.5) that

$$\mathbb{E}_{\mathbb{P}}\big[(\mathscr{Y}^n)^{t,\omega}_{\gamma}\big]\le\mathbb{E}_{\mathbb{P}}\Big[\big(\mathscr{Y}^{n+1}\big)^{t,\omega}_{\widetilde{\gamma}}-J_{\gamma,T}\Big]+\widehat{\rho}_0(\delta)\le\mathscr{Z}^{n+1}_t(\omega)-\widetilde{\Delta}^U+2\widehat{\rho}_0(\delta).$$

Then taking supremum over $(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t$ on the left-hand-side leads to that $\mathscr{Z}^n_t(\omega)\le\mathscr{Z}^{n+1}_t(\omega)-\widetilde{\Delta}^U+2\widehat{\rho}_0(\delta)$. $\square$

Proof of Proposition 5.3: 1) Let $n\in\mathbb{N}$. Lemma 5.1 and Proposition 4.2 show that $Z^{n,k}$, $k\in\mathbb{N}$ are $\mathbf{F}-$adapted processes with all continuous paths. For any $(t,\omega)\in[0,T]\times\Omega$, as $k\to\infty$ in (5.4), the continuity of U implies that

$$\lim_{k\to\infty}Z^{n,k}_t(\omega)=\mathscr{Z}^n_t(\omega).\tag{6.65}$$

Then the $\mathbf{F}-$adaptedness of $\{Z^{n,k}\}_{k\in\mathbb{N}}$ shows that process $\mathscr{Z}^n$ is also $\mathbf{F}-$adapted.

Given $(s,\omega)\in[0,T]\times\Omega$, letting $t\to s$ in (5.4), we can deduce from the continuity of processes U, $\{Z^{n,k}\}_{k\in\mathbb{N}}$ that

$$\begin{aligned}Z^{n,k}_s(\omega)-\widehat{\rho}_0(2^{1-k})-U\big((\wp_n(\omega)+2^{1-k})\wedge s,\omega\big)+U(\wp_n(\omega)\wedge s,\omega)&\le\varliminf_{t\to s}\mathscr{Z}^n_s(\omega)\le\varlimsup_{t\to s}\mathscr{Z}^n_s(\omega)\\&\le Z^{n,k}_s(\omega)+2\widehat{\rho}_0(2^{1-k})-U\big((\wp_n(\omega)+2^{1-k})\wedge s,\omega\big)+U(\wp_n(\omega)\wedge s,\omega),\quad\forall\,k\in\mathbb{N}.\end{aligned}$$

As $k\to\infty$, (6.65) and the continuity of U imply that $\lim\limits_{t\to s}\mathscr{Z}^n_t(\omega)=\lim\limits_{k\to\infty}Z^{n,k}_s(\omega)=\mathscr{Z}^n_s(\omega)$. Hence, the process $\mathscr{Z}^n$ has all continuous paths.

2) Fix $(t,\omega)\in[0,T]\times\Omega$. For any integers $n<m$, adding (5.5) up from $i=n$ to $i=m-1$ shows that

$$-2\sum_{i=n}^{m-1}\widehat{\rho}_0\big(\tfrac{2T}{i+3}\big)\ \le\ \mathscr{Z}^m_t(\omega)-\mathscr{Z}^n_t(\omega)-U\big(\wp_m(\omega)\wedge t,\omega\big)+U\big(\wp_n(\omega)\wedge t,\omega\big)\le\sum_{i=n}^{m-1}\widehat{\rho}_0\big(\tfrac{2T}{i+3}\big).\tag{6.66}$$

Since $\widehat{\mathfrak{p}}_1>1$ by (P2), (2.4) gives that $\sum_{i=0}^{\infty}\widehat{\rho}_0\big(\frac{2T}{i+3}\big)\le\sum_{i=0}^{n_0-1}\widehat{\rho}_0\big(\frac{2T}{i+3}\big)+\widehat{\mathfrak{C}}\sum_{i=n_0}^{\infty}\big(\frac{2T}{i+3}\big)^{\widehat{\mathfrak{p}}_1}<\infty$, where $n_0:=1+\lfloor(2T-3)^+\rfloor$. Then we see from the continuity of U and (6.66) that $\big\{\mathscr{Z}^n_t(\omega)\big\}_{n\in\mathbb{N}}$ is a Cauchy sequence of $\mathbb{R}$. Let $\mathscr{Z}_t(\omega)$ be the limit of $\big\{\mathscr{Z}^n_t(\omega)\big\}_{n\in\mathbb{N}}$, i.e. $\mathscr{Z}_t(\omega):=\lim\limits_{n\to\infty}\mathscr{Z}^n_t(\omega)$. As $\lim\limits_{m\to\infty}\uparrow\tau_m(\omega)=\tau_0(\omega)$, Proposition 5.1 (1) shows that $\lim\limits_{m\to\infty}\uparrow\wp_m(\omega)=\tau_0(\omega)$. Letting $m\to\infty$ in (6.66) and using the continuity of U yield (5.6).

3a) *Let us now show the first inequality of* (5.7).

Clearly, the $\mathbf{F}-$adaptedness of $\{\mathscr{Z}^n\}_{n\in\mathbb{N}}$ implies that of $\mathscr{Z}$ and the boundedness of $\{\mathscr{Z}^n\}_{n\in\mathbb{N}}$ by M_0 implies that of $\mathscr{Z}$. Similar to the argument used in part 1), letting $t\to s$ in (5.6), we can deduce from the continuity of processes $\{\mathscr{Z}^n\}_{n\in\mathbb{N}}$, U and $\lim\limits_{n\to\infty}\uparrow\wp_n=\tau_0$ that the process $\mathscr{Z}$ has all continuous paths.

Let $(t,\omega)\in[0,T]\times\Omega$. Given $\varepsilon>0$, there exists $(\mathbb{P}_\varepsilon,\gamma_\varepsilon)\in\mathcal{P}_t\times\mathcal{T}^t$ such that $\sup\limits_{(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t}\mathbb{E}_\mathbb{P}\Big[\widehat{\mathscr{Y}}^{\,t,\omega}_{\gamma}\Big]\le\mathbb{E}_{\mathbb{P}_\varepsilon}\Big[\widehat{\mathscr{Y}}^{\,t,\omega}_{\gamma_\varepsilon}\Big]+\varepsilon$. Since $\lim\limits_{n\to\infty}\uparrow\tau_n=\tau_0$, one can deduce from the continuity of U that

$$\lim_{n\to\infty}\mathscr{Y}^n_{t'}(\omega')=\widehat{\mathscr{Y}}_{t'}(\omega'),\quad\forall\,(t',\omega')\in[0,T]\times\Omega.\tag{6.67*}$$

It follows that $\lim\limits_{n\to\infty}(\mathscr{Y}^n)^{t,\omega}_{\gamma_\varepsilon}(\widetilde{\omega})=\lim\limits_{n\to\infty}\mathscr{Y}^n\big(\gamma_\varepsilon(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)=\widehat{\mathscr{Y}}\big(\gamma_\varepsilon(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)=\widehat{\mathscr{Y}}^{\,t,\omega}_{\gamma_\varepsilon}(\widetilde{\omega})$, $\forall\,\widetilde{\omega}\in\Omega^t$. As $\mathscr{Y}^n$'s are all bounded by M_0, applying the bounded convergence theorem yields that

$$\sup_{(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t}\mathbb{E}_\mathbb{P}\Big[\widehat{\mathscr{Y}}^{\,t,\omega}_{\gamma}\Big]\le\mathbb{E}_{\mathbb{P}_\varepsilon}\Big[\widehat{\mathscr{Y}}^{\,t,\omega}_{\gamma_\varepsilon}\Big]+\varepsilon=\lim_{n\to\infty}\mathbb{E}_{\mathbb{P}_\varepsilon}\big[(\mathscr{Y}^n)^{t,\omega}_{\gamma_\varepsilon}\big]+\varepsilon\le\lim_{n\to\infty}\mathscr{Z}^n_t(\omega)+\varepsilon=\mathscr{Z}_t(\omega)+\varepsilon.$$

Then letting $\varepsilon\to0$ leads to that $\mathscr{Z}_t(\omega)\ge\sup\limits_{(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t}\mathbb{E}_\mathbb{P}\Big[\widehat{\mathscr{Y}}^{\,t,\omega}_{\gamma}\Big]\ge\sup\limits_{\mathbb{P}\in\mathcal{P}_t}\mathbb{E}_\mathbb{P}\Big[\widehat{\mathscr{Y}}^{\,t,\omega}_{t}\Big]=\widehat{\mathscr{Y}}_t(\omega)$, where we used the $\mathbf{F}-$adaptedness of $\widehat{\mathscr{Y}}$ and (2.6) in the last equality.

3b) *Let $(t,\omega)\in[0,T]\times\Omega$. We verify the third equality of* (5.7) *by two cases.*

If $\tau_0(\omega)=T$, (6.62) and the continuity of U imply that

$$\mathscr{Z}_T(\omega)=\lim_{n\to\infty}\mathscr{Z}^n_T(\omega)=\lim_{n\to\infty}\sup_{\mathbb{P}\in\mathcal{P}_t}\mathbb{E}_\mathbb{P}\big[(\mathscr{Y}^n)^{T,\omega}_T\big]=\lim_{n\to\infty}\mathscr{Y}^n_T(\omega)=\lim_{n\to\infty}U\big(\wp_n(\omega),\omega\big)=U\big(\tau_0(\omega),\omega\big).$$

Suppose next that $\tau_0(\omega)<T$. By the definition of $\tau_0(\omega)$, the set $\{t\in[0,T]:\mathscr{X}_t(\omega)\le0\}$ is not empty. So Proposition 5.1 shows that $\wp_n(\omega)<\tau_0(\omega)$.

Let $t\in[\tau_0(\omega),T]$ and $n\in\mathbb{N}$. As $t_n:=\wp_n(\omega)<\tau_0(\omega)\le t$, Lemma A.1 implies that $\wp_n(\omega\otimes_t\Omega^t)=\wp_n(\omega)=t_n$. Let $\gamma\in\mathcal{T}^t$. Since U is an $\mathbf{F}-$adapted process by (A1) and (2.3), one has $U_{t_n}\in\mathcal{F}_{t_n}\subset\mathcal{F}_t$. Given $\widetilde{\omega}\in\Omega^t$, using (2.6) with $(t,s,\eta)=(0,t,U_{t_n})$ shows that $U(t_n,\omega\otimes_t\widetilde{\omega})=U(t_n,\omega)$. Then we can deduce from $\gamma(\widetilde{\omega})\ge t>t_n=\wp_n(\omega\otimes_t\widetilde{\omega})$ that

$$(\mathscr{Y}^n)^{t,\omega}_\gamma(\widetilde{\omega})=\mathscr{Y}^n\big(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)=U\big(\wp_n(\omega\otimes_t\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)=U(t_n,\omega\otimes_t\widetilde{\omega})=U(t_n,\omega),$$

which leads to that $\mathscr{Z}^n(t,\omega)=\sup\limits_{(\mathbb{P},\gamma)\in\mathcal{P}_t\times\mathcal{T}^t}\mathbb{E}_\mathbb{P}\big[(\mathscr{Y}^n)^{t,\omega}_\gamma\big]=U(t_n,\omega)=U\big(\wp_n(\omega),\omega\big)$. Letting $n\to\infty$, we obtain from the continuity of U that $\mathscr{Z}\big(t,\omega\big)=U\big(\tau_0(\omega),\omega\big)$.

4) By (3.3) and the continuity of $\mathscr{Z}$ obtained in part 3a), $D_t:=\mathscr{Z}_t-\widehat{\mathscr{Y}}_t\ge0$, $t\in[0,T]$ is an $\mathbf{F}-$adapted process whose paths are all continuous except a possible negative jump at τ_0. In particular, each path of D is lower-semicontinuous and right-continuous. It follows that γ_* is an $\mathbf{F}-$stopping time (see Lemma A.13 in the ArXiv version of [6] for a proof).

As $\mathscr{Z}_t=U_{\tau_0}=\widehat{\mathscr{Y}}_t$, $\forall\,t\in[\tau_0,T]$ by (5.7), one can deduce that $\gamma_*=\gamma_*\wedge\tau_0=\inf\big\{t\in[0,\tau_0):\mathscr{Z}_t=\widehat{\mathscr{Y}}_t\big\}\wedge\tau_0=\inf\big\{t\in[0,\tau_0):\mathscr{Z}_t=\mathscr{Y}_t\big\}\wedge\tau_0=\inf\{t\in[0,\tau_0):\mathscr{Z}_t=L_t\}\wedge\tau_0$. □

6.4 Proof of Theorem 3.1

For any $m\in\mathbb{N}$, applying Theorem 4.1 with $(Y,\wp)=(Y^{m,m},\wp^{m,m})$ shows that there exists a $\mathbb{P}_m\in\mathcal{P}$ such that

$$Z^{m,m}_0=\mathbb{E}_{\mathbb{P}_m}\Big[Z^{m,m}_{\nu_m\wedge\zeta}\Big],\quad\forall\,\zeta\in\mathcal{T},\tag{6.68}$$

where $\nu_m:=\inf\big\{t\in[0,T]:Z^{m,m}_t=\widehat{Y}^{m,m}_t\big\}\in\mathcal{T}$. By (P1), $\{\mathbb{P}_m\}_{m\in\mathbb{N}}$ has a weakly convergent sequence $\{\mathbb{P}_{m_j}\}_{i\in\mathbb{N}}$ with limit $\mathbb{P}_*$.

1) *First, we use* (5.4), (5.6) *and similar arguments to those proving Theorem 4.1 to show that*

$$\mathscr{Z}_0\le\mathbb{E}_{\mathbb{P}_*}\Big[\varlimsup_{n\to\infty}\varlimsup_{i\to\infty}\varlimsup_{\ell\to\infty}\mathscr{Z}_{\zeta_{i,\ell}\wedge\wp_n}\Big],\tag{6.69}$$

where $\zeta_{i,\ell}:=\inf\big\{t\in[0,T]:Z^{\ell,\ell}_t\le L_t+1/i\big\}\wedge T$. This part is relatively lengthy, we will split it into several steps.

1a) *We start with an auxiliary inequality: for any $n,k\in\mathbb{N}$ with $k\ge n$ and $\omega\in\Omega$,*

$$\big|Z^{k,k}_t(\omega)-\mathscr{Z}_t(\omega)\big|\le\overline{\varepsilon}_k:=2\widehat{\rho}_0\big(2^{1-k}\big)+2\sum_{i=k}^\infty\widehat{\rho}_0\big(\tfrac{2T}{i+3}\big),\quad\forall\,t\in[0,\wp_n(\omega)].\tag{6.70}$$

Let $n,k\in\mathbb{N}$ with $k\ge n$ and let $\omega\in\Omega$. For any $t\in[0,T]$, we see from (5.4) and (5.6) that $-2\widehat{\rho}_0\big(2^{1-k}\big)\le Z^{k,k}_t(\omega)-\mathscr{Z}^k_t(\omega)-U\big((\wp_k(\omega)+2^{1-k})\wedge t,\omega\big)+U\big(\wp_k(\omega)\wedge t,\omega\big)\le\widehat{\rho}_0\big(2^{1-k}\big)$ and that $-\sum_{i=k}^\infty\widehat{\rho}_0(\frac{2T}{i+3})\le\mathscr{Z}^k_t(\omega)-\mathscr{Z}_t(\omega)-U\big(\wp_k(\omega)\wedge t,\omega\big)+U\big(\tau_0(\omega)\wedge t,\omega\big)\le 2\sum_{i=k}^\infty\widehat{\rho}_0(\frac{2T}{i+3})$. Adding them together yields that

$$-\overline{\varepsilon}_k\le Z^{k,k}_t(\omega)-\mathscr{Z}_t(\omega)-U\big((\wp_k(\omega)+2^{1-k})\wedge t,\omega\big)+U\big(\tau_0(\omega)\wedge t,\omega\big)\le\overline{\varepsilon}_k,\quad\forall\, t\in[0,T].\tag{6.71}$$

In particular, for any $t\in[0,\wp_n(\omega)]$, since $t\le\wp_n(\omega)\le\wp_k(\omega)\le\tau_0(\omega)$ by Proposition 5.1 (1), one has $U\big((\wp_k(\omega)+2^{1-k})\wedge t,\omega\big)=U\big(\tau_0(\omega)\wedge t,\omega\big)=U(t,\omega)$. Then (6.70) directly follows from (6.71).

Now, fix integers $1\le n<i<\ell<\alpha$ such that $\overline{\varepsilon}_\ell\le\frac{1}{2i}$ and fix $j\in\mathbb{N}$ such that $m_j\ge\alpha$. Since Lemma 5.1, Proposition 4.2, (A1) and (2.3) show that $Z^{\ell,\ell}-L$ is an $\mathbf{F}-$adapted process with all continuous paths,

$$\zeta^\alpha_{i,\ell}:=\inf\big\{t\in[0,T]:Z^{\ell,\ell}_t\le L_t+1/i+1/\alpha\big\}\wedge T\ \text{ defines an }\mathbf{F}\text{−stopping time.}\tag{6.72}$$

Similar to ν_n in (6.29), $\widehat{\zeta}^\alpha_{i,\ell}:=\inf\big\{t\in[0,T]:Z^{\ell,\ell}_t\le\widehat{Y}^{\ell,\ell}_t+1/i+1/\alpha\big\}$ is also an $\mathbf{F}-$stopping time satisfying

$$\widehat{\zeta}^\alpha_{i,\ell}\wedge\wp_n=\zeta^\alpha_{i,\ell}\wedge\wp_n\le\nu_{m_j}\wedge\wp_n.\tag{6.73*}$$

Then applying (6.70) with $(k,t)=(m_j,0)$, $(k,t)=\big(m_j,\widehat{\zeta}^\alpha_{i,\ell}\wedge\wp_n\big)$ and $(k,t)=\big(\ell,\widehat{\zeta}^\alpha_{i,\ell}\wedge\wp_n\big)$ respectively as well as applying (6.68) with $(m,\zeta)=\big(m_j,\widehat{\zeta}^\alpha_{i,\ell}\wedge\wp_n\big)$, we obtain

$$\mathscr{Z}_0-\overline{\varepsilon}_{m_j}\le Z^{m_j,m_j}_0=\mathbb{E}_{\mathbb{P}_{m_j}}\Big[Z^{m_j,m_j}_{\nu_{m_j}\wedge\widehat{\zeta}^\alpha_{i,\ell}\wedge\wp_n}\Big]=\mathbb{E}_{\mathbb{P}_{m_j}}\Big[Z^{m_j,m_j}_{\widehat{\zeta}^\alpha_{i,\ell}\wedge\wp_n}\Big]\le\mathbb{E}_{\mathbb{P}_{m_j}}\Big[\mathscr{Z}_{\widehat{\zeta}^\alpha_{i,\ell}\wedge\wp_n}\Big]+\overline{\varepsilon}_{m_j}\le\mathbb{E}_{\mathbb{P}_{m_j}}\Big[Z^{\ell,\ell}_{\widehat{\zeta}^\alpha_{i,\ell}\wedge\wp_n}\Big]+\overline{\varepsilon}_{m_j}+\overline{\varepsilon}_\ell.\tag{6.74}$$

1b) *Before sending j to ∞ in order to approximate the distribution $\mathbb{P}_*$ in* (6.35)*, we need to approach $\big\{\widehat{\zeta}^\alpha_{i,\ell}\big\}_{\alpha\in\mathbb{N}}$ by a sequence $\big\{\theta^\alpha_{i,\ell}\big\}_{\alpha\in\mathbb{N}}$ of Lipschitz continuous random variables and estimate the expected difference $\mathbb{E}_{\mathbb{P}_{m_j}}\Big[\Big|Z^{\ell,\ell}_{\widehat{\zeta}^\alpha_{i,\ell}\wedge\wp_n}-Z^{\ell,\ell}_{\theta^\alpha_{i,\ell}\wedge\wp_n}\Big|\Big]$.*

Recall from Lemma 5.1 and the remark following it that $Y^{\ell,\ell}$ is uniformly continuous on $[0,T]\times\Omega$ with respect to the modulus of continuity function $\rho_{\ell,\ell}$ and that $\wp^{\ell,\ell}$ is a Lipschitz continuous stopping time on Ω with coefficient κ_ℓ. Replacing $(Z,\widehat{Y},\nu_n)$ by $\big(Z^{\ell,\ell},\widehat{Y}^{\ell,\ell},\widehat{\zeta}^\alpha_{i,\ell}\big)$ in the arguments that lead to (6.38), we can find an open subset $\Omega^\alpha_{i,\ell}$ of Ω and a Lipschitz continuous random variable $\theta^\alpha_{i,\ell}:\Omega\to[0,T]$ such that

$$\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{P}\big((\Omega^\alpha_{i,\ell})^c\big)\le 2^{-\alpha},\quad \widehat{\zeta}^{\alpha-1}_{i,\ell}-2^{-\alpha}<\theta^\alpha_{i,\ell}<\widehat{\zeta}^{\alpha+1}_{i,\ell}+2^{-\alpha}\ \text{ on }\ \Omega^\alpha_{i,\ell}.\tag{6.75}$$

Given $\omega\in\widehat{\Omega}^{\alpha-1}_{i,\ell}\cap\widehat{\Omega}^{\alpha+1}_{i,\ell}$, since $\theta^{\alpha-1}_{i,\ell}-2^{-\alpha+1}<\widehat{\zeta}^\alpha_{i,\ell}<\theta^{\alpha+1}_{i,\ell}+2^{-\alpha-1}$, (2.5) and an analogy to (6.39) imply that $t:=\theta^\alpha_{i,\ell}(\omega)\wedge\widehat{\zeta}^\alpha_{i,\ell}(\omega)\wedge\wp_n$ and $s:=\Big(\theta^\alpha_{i,\ell}(\omega)\vee\widehat{\zeta}^\alpha_{i,\ell}(\omega)\Big)\wedge\wp_n$ satisfy

$$s-t=\big|\widehat{\zeta}^\alpha_{i,\ell}(\omega)\wedge\wp_n(\omega)-\theta^\alpha_{i,\ell}(\omega)\wedge\wp_n(\omega)\big|\le\big|\widehat{\zeta}^\alpha_{i,\ell}(\omega)-\theta^\alpha_{i,\ell}(\omega)\big|<|\theta^{\alpha-1}_{i,\ell}(\omega)-\theta^\alpha_{i,\ell}(\omega)|+|\theta^{\alpha+1}_{i,\ell}(\omega)-\theta^\alpha_{i,\ell}(\omega)|+2^{-\alpha+1}:=\delta^\alpha_{i,\ell}(\omega).$$

Set $\phi^\alpha_{i,\ell}(\omega):=(1+\kappa_\ell)\Big(\big(\delta^\alpha_{i,\ell}(\omega)\big)^{\frac{q_1}{2}}+\phi^\omega_T\big(\delta^\alpha_{i,\ell}(\omega)\big)\Big)$. An application of (4.5) to $Z=Z^{\ell,\ell}$ shows that

$$\begin{aligned}&\big|Z^{\ell,\ell}_{\theta^\alpha_{i,\ell}\wedge\wp_n}(\omega)-Z^{\ell,\ell}_{\widehat{\zeta}^\alpha_{i,\ell}\wedge\wp_n}(\omega)\big|=\big|Z^{\ell,\ell}(t,\omega)-Z^{\ell,\ell}(s,\omega)\big|\\&\quad\le 2C_\varrho M_0\Big(\big(\delta^\alpha_{i,\ell}(\omega)\big)^{\frac{q_1}{2}}\vee\big(\delta^\alpha_{i,\ell}(\omega)\big)^{q_2-\frac{q_1}{2}}\Big)+\widehat{\rho}_{\ell,\ell}\big(\delta^\alpha_{i,\ell}(\omega)\big)+\widehat{\rho}_{\ell,\ell}\big(\phi^\alpha_{i,\ell}(\omega)\big)\vee\widehat{\widehat{\rho}}_{\ell,\ell}\big(\phi^\alpha_{i,\ell}(\omega)\big):=\xi^\alpha_{i,\ell}(\omega).\end{aligned}$$

As $Z^{\ell,\ell}$ is bounded by M_0, (6.74) and (6.75) imply that

$$\begin{aligned}\mathscr{Z}_0-2\overline{\varepsilon}_{m_j}-\overline{\varepsilon}_\ell&\le\mathbb{E}_{\mathbb{P}_{m_j}}\Big[Z^{\ell,\ell}_{\theta^\alpha_{i,\ell}\wedge\wp_n}\Big]+\mathbb{E}_{\mathbb{P}_{m_j}}\Big[\Big|Z^{\ell,\ell}_{\widehat{\zeta}^\alpha_{i,\ell}\wedge\wp_n}-Z^{\ell,\ell}_{\theta^\alpha_{i,\ell}\wedge\wp_n}\Big|\Big]\\&\le\mathbb{E}_{\mathbb{P}_{m_j}}\Big[Z^{\ell,\ell}_{\theta^\alpha_{i,\ell}\wedge\wp_n}\Big]+\mathbb{E}_{\mathbb{P}_{m_j}}\Big[\mathbf{1}_{\widehat{\Omega}^{\alpha-1}_{i,\ell}\cap\widehat{\Omega}^{\alpha+1}_{i,\ell}}(\xi^\alpha_{i,\ell}\wedge 2M_0)\Big]+2M_0\mathbb{P}_{m_j}\Big(\big(\widehat{\Omega}^{\alpha-1}_{i,\ell}\big)^c\cup\big(\widehat{\Omega}^{\alpha+1}_{i,\ell}\big)^c\Big)\\&\le\mathbb{E}_{\mathbb{P}_{m_j}}\Big[Z^{\ell,\ell}_{\theta^\alpha_{i,\ell}\wedge\wp_n}+(\xi^\alpha_{i,\ell}\wedge 2M_0)\Big]+5M_0 2^{-\alpha}.\end{aligned}\tag{6.76}$$

The random variables $\theta^{\alpha-1}_{i,\ell},\theta^\alpha_{i,\ell},\theta^{\alpha+1}_{i,\ell}$ are Lipschitz continuous on Ω, so is $\delta^\alpha_{i,\ell}$. Similar to (6.41), one can show that $\omega\to\phi^\omega_T\big(\delta^\alpha_{i,\ell}(\omega)\big)$ is also a continuous random variable on Ω, which together with the Lipschitz continuity of $\delta^\alpha_{i,\ell}$

implies that $\phi^\alpha_{i,\ell}$ and thus $\xi^\alpha_{i,\ell}$ are also continuous random variables on Ω. Analogous to (6.42), we can deduce from the Lipschitz continuity of random variable $\theta^\alpha_{i,\ell}\wedge\wp_n$ and the continuity of process Z that $Z^\ell_{\theta^\alpha_{i,\ell}\wedge\wp_n}$ is a continuous random variable on Ω.

As Proposition 5.3 (2) shows that

$$\lim_{m\to\infty}\downarrow\overline{\varepsilon}_m=0, \tag{6.77}$$

letting $j\to\infty$ in (6.76), we see from the continuity of random variables $Z^{\ell,\ell}_{\theta^\alpha_{i,\ell}\wedge\wp_n}$ and $\xi^\alpha_{i,\ell}$ that

$$\mathscr{Z}_0\le\mathbb{E}_{\mathbb{P}_*}\Big[Z^{\ell,\ell}_{\theta^\alpha_{i,\ell}\wedge\wp_n}+(\xi^\alpha_{i,\ell}\wedge 2M_0)\Big]+\overline{\varepsilon}_\ell+5M_0 2^{-\alpha}. \tag{6.78}$$

1c) *Next, we will use the convergence of $\theta^\alpha_{i,\ell}$ to $\widehat{\zeta}_{i,\ell}$, the continuity of $Z^{\ell,\ell}$ as well as (6.70) to derive (6.69).*

Since the continuity of $Z^{\ell,\ell}-\widehat{Y}^{\ell,\ell}$ implies that

$$\lim_{\alpha\to\infty}\uparrow\widehat{\zeta}^\alpha_{i,\ell}=\widehat{\zeta}_{i,\ell}:=\inf\big\{t\in[0,T]:Z^{\ell,\ell}_t\le\widehat{Y}^{\ell,\ell}_t+1/i\big\}\in\mathcal{T}, \tag{6.79*}$$

using an analogy to (6.44) we can deduce from (6.75) and the Borel-Cantelli Lemma that $\lim\limits_{\alpha\to\infty}\theta^\alpha_{i,\ell}=\widehat{\zeta}_{i,\ell}$, $\mathbb{P}_*-$a.s. It follows that $\lim\limits_{\alpha\to\infty}\delta^\alpha_{i,\ell}=0$, $\mathbb{P}_*-$a.s. and thus $\lim\limits_{\alpha\to\infty}\xi^\alpha_{i,\ell}=0$, $\mathbb{P}_*-$a.s. As Proposition 4.2 shows that $Z^{\ell,\ell}$ is an $\mathbf{F}-$adapted process bounded by M_0 that has all continuous paths, letting $\alpha\to\infty$ in (6.78) we see from the bounded dominated convergence theorem that

$$\mathscr{Z}_0\le\mathbb{E}_{\mathbb{P}_*}\Big[Z^{\ell,\ell}_{\widehat{\zeta}_{i,\ell}\wedge\wp_n}\Big]+\overline{\varepsilon}_\ell. \tag{6.80}$$

Similar to $\zeta^\alpha_{i,\ell}$ in (6.72), $\zeta_{i,\ell}$ is an $\mathbf{F}-$stopping time satisfying $\zeta_{i,\ell}\wedge\wp_n=\widehat{\zeta}_{i,\ell}\wedge\wp_n$. Applying (6.70) with $(k,t)=\big(\ell,\zeta_{i,\ell}\wedge\wp_n\big)$ and using (6.80) yield that $\mathscr{Z}_0\le\mathbb{E}_{\mathbb{P}_*}\Big[Z^{\ell,\ell}_{\zeta_{i,\ell}\wedge\wp_n}\Big]+\overline{\varepsilon}_\ell\le\mathbb{E}_{\mathbb{P}_*}\Big[\mathscr{Z}_{\zeta_{i,\ell}\wedge\wp_n}\Big]+2\overline{\varepsilon}_\ell$. Since Proposition 5.3 (3) shows that $\mathscr{Z}$ is bounded by M_0, letting $\ell\to\infty$, using the Fatou's Lemma and (6.77) yield that $\mathscr{Z}_0\le\overline{\lim\limits_{\ell\to\infty}}\,\mathbb{E}_{\mathbb{P}_*}\Big[\mathscr{Z}_{\zeta_{i,\ell}\wedge\wp_n}\Big]\le\mathbb{E}_{\mathbb{P}_*}\Big[\overline{\lim\limits_{\ell\to\infty}}\,\mathscr{Z}_{\zeta_{i,\ell}\wedge\wp_n}\Big]$. Similarly, letting $i\to\infty$ and then letting $n\to\infty$, we derive (6.69) from Fatou's Lemma again.

2) *In the second part, we show that for any $i\in\mathbb{N}$*

$$\gamma_i\le\underline{\lim_{\ell\to\infty}}\,\zeta_{i,\ell}\le\overline{\lim_{\ell\to\infty}}\,\zeta_{i,\ell}\le\gamma_{2i}\quad\text{holds on }\Omega, \tag{6.81}$$

where $\gamma_i:=\inf\big\{t\in[0,T]:\mathscr{Z}_t\le L_t+1/i\big\}\wedge T$.

Fix $i\in\mathbb{N}$. Since Proposition 5.3 (3), (A1) and (2.3) show that $\mathscr{Z}-L$ is an $\mathbf{F}-$adapted process with all continuous paths, γ_i is an $\mathbf{F}-$stopping time that satisfies

$$\gamma_i=\lim_{h\to\infty}\uparrow\gamma^h_i, \tag{6.82*}$$

where $\gamma^h_i:=\inf\big\{t\in[0,T]:\mathscr{Z}_t\le L_t+1/i+1/h\big\}\wedge T\in\mathcal{T}$.

Fix $\omega\in\Omega$ and define $\phi^\omega_U(x):=\sup\big\{|U_{r'}(\omega)-U_r(\omega)|:r,r'\in[0,T],\ 0\le|r'-r|\le x\big\}$, $\forall\,x\in[0,T]$. For any $\ell\in\mathbb{N}$, since (2.5) implies that $\big|U\big((\wp_\ell(\omega)+2^{1-\ell})\wedge\zeta_{i,\ell}(\omega),\omega\big)-U\big(\tau_0(\omega)\wedge\zeta_{i,\ell}(\omega),\omega\big)\big|\le\phi^\omega_U\big(\big|(\wp_\ell(\omega)+2^{1-\ell})\wedge\zeta_{i,\ell}(\omega)-\tau_0(\omega)\wedge\zeta_{i,\ell}(\omega)\big|\big)\le\phi^\omega_U\big(\big|(\wp_\ell(\omega)+2^{1-\ell})\wedge T-\tau_0(\omega)\big|\big)$, applying (6.71) with $(k,t)=\big(\ell,\zeta_{i,\ell}(\omega)\big)$ implies that

$$\big|Z^{\ell,\ell}\big(\zeta_{i,\ell}(\omega),\omega\big)-\mathscr{Z}\big(\zeta_{i,\ell}(\omega),\omega\big)\big|\le\overline{\varepsilon}_\ell+\phi^\omega_U\big(\big|(\wp_\ell(\omega)+2^{1-\ell})\wedge T-\tau_0(\omega)\big|\big),\quad\forall\,\ell\in\mathbb{N}. \tag{6.83}$$

As $\lim\limits_{\ell\to\infty}\uparrow\wp_\ell(\omega)=\tau_0(\omega)$ by Proposition 5.1 (1), the uniform continuity of the path $U_\cdot(\omega)$ implies that

$$\lim_{\ell\to\infty}\phi^\omega_U\big(\big|(\wp_\ell(\omega)+2^{1-\ell})\wedge T-\tau_0(\omega)\big|\big)=0. \tag{6.84}$$

To see the first inequality of (6.81), we assume without loss of generality that $\underline{\lim\limits_{\ell\to\infty}}\,\zeta_{i,\ell}(\omega)<T$. There exists a subsequence $\{\ell_\lambda=\ell_\lambda(i,\omega)\}_{\lambda\in\mathbb{N}}$ of $\mathbb{N}$ such that $\lim\limits_{\lambda\to\infty}\zeta_{i,\ell_\lambda}(\omega)=\underline{\lim\limits_{\ell\to\infty}}\,\zeta_{i,\ell}(\omega)<T$.

Let $h\in\mathbb{N}$. Since $\lim\limits_{\ell\to\infty}\downarrow\overline{\varepsilon}_\ell=0$ and because of (6.84), there exists a $\widehat{\lambda}_h=\widehat{\lambda}_h(i,\omega)\in\mathbb{N}$ such that for any integer $\lambda\geq\widehat{\lambda}_h$, one has $\zeta_{i,\ell_\lambda}(\omega)<T$ and $\overline{\varepsilon}_{\ell_\lambda}+\phi_U^\omega\big(\big|(\wp_{\ell_\lambda}(\omega)+2^{1-\ell_\lambda})\wedge T-\tau_0(\omega)\big|\big)\leq 1/h$. Given $\lambda\in\mathbb{N}$ with $\lambda\geq\widehat{\lambda}_h$, as $\zeta_{i,\ell_\lambda}(\omega)<T$, the set $\big\{t\in[0,T]\colon Z_t^{\ell_\lambda,\ell_\lambda}(\omega)\leq L_t(\omega)+1/i\big\}$ is not empty. So the continuity of the path $Z_\cdot^{\ell_\lambda,\ell_\lambda}(\omega)-L_\cdot(\omega)$ implies that $Z^{\ell_\lambda,\ell_\lambda}\big(\zeta_{i,\ell_\lambda}(\omega),\omega\big)\leq L\big(\zeta_{i,\ell_\lambda}(\omega),\omega\big)+1/i$. Applying (6.83) with $\ell=\ell_\lambda$ yields that

$$\mathscr{Z}\big(\zeta_{i,\ell_\lambda}(\omega),\omega\big)\leq Z^{\ell_\lambda,\ell_\lambda}\big(\zeta_{i,\ell_\lambda}(\omega),\omega\big)+\overline{\varepsilon}_{\ell_\lambda}+\phi_U^\omega\big(\big|(\wp_{\ell_\lambda}(\omega)+2^{1-\ell_\lambda})\wedge T-\tau_0(\omega)\big|\big)\leq L\big(\zeta_{i,\ell_\lambda}(\omega),\omega\big)+1/i+1/h,$$

which shows that $\gamma_i^h(\omega)\leq\zeta_{i,\ell_\lambda}(\omega)$. As $\lambda\to\infty$, we obtain $\gamma_i^h(\omega)\leq\lim\limits_{\lambda\to\infty}\zeta_{i,\ell_\lambda}(\omega)=\varliminf\limits_{\ell\to\infty}\zeta_{i,\ell}(\omega)$. Then letting $h\to\infty$ and using (6.82) yield that $\gamma_i(\omega)=\lim\limits_{h\to\infty}\uparrow\gamma_i^h(\omega)\leq\varliminf\limits_{\ell\to\infty}\zeta_{i,\ell}(\omega)$.

As to the third inequality of (6.81), we assume without loss of generality that $\gamma_{2i}(\omega)<T$, or equivalently, the set $\big\{t\in[0,T]\colon\mathscr{Z}_t(\omega)\leq L_t(\omega)+\frac{1}{2i}\big\}$ is not empty. Then one can deduce from the continuity of the path $\mathscr{Z}_\cdot(\omega)-L_\cdot(\omega)$ that

$$\mathscr{Z}\big(\gamma_{2i}(\omega),\omega\big)\leq L\big(\gamma_{2i}(\omega),\omega\big)+\frac{1}{2i}.\tag{6.85}$$

Applying (6.71) with $(k,t)=\big(\ell,\gamma_{2i}(\omega)\big)$ and using a similar argument to the one that leads to (6.83) yield that

$$\big|Z^{\ell,\ell}\big(\gamma_{2i}(\omega),\omega\big)-\mathscr{Z}\big(\gamma_{2i}(\omega),\omega\big)\big|\leq\overline{\varepsilon}_\ell+\phi_U^\omega\big(\big|(\wp_\ell(\omega)+2^{1-\ell})\wedge T-\tau_0(\omega)\big|\big).\tag{6.86}$$

For any $\ell\in\mathbb{N}$ such that $\overline{\varepsilon}_\ell+\phi_U^\omega\big(\big|(\wp_\ell(\omega)+2^{1-\ell})\wedge T-\tau_0(\omega)\big|\big)\leq\frac{1}{2i}$, (6.85) and (6.86) imply that $Z^{\ell,\ell}\big(\gamma_{2i}(\omega),\omega\big)\leq\mathscr{Z}\big(\gamma_{2i}(\omega),\omega\big)+\frac{1}{2i}\leq L\big(\gamma_{2i}(\omega),\omega\big)+1/i$, which shows that $\zeta_{i,\ell}(\omega)\leq\gamma_{2i}(\omega)$. As $\ell\to\infty$, we obtain $\varlimsup\limits_{\ell\to\infty}\zeta_{i,\ell}(\omega)\leq\gamma_{2i}(\omega)$.

3) *Finally, we show that* $\varlimsup\limits_{n\to\infty}\varlimsup\limits_{i\to\infty}\varlimsup\limits_{\ell\to\infty}\mathscr{Z}(\zeta_{i,\ell}(\omega)\wedge\wp_n(\omega),\omega)=\mathscr{Z}(\gamma_*(\omega),\omega)$, $\forall\,\omega\in\Omega$. *The conclusion thus follows.*

Let $1\leq n<i$ and $\omega\in\Omega$. We set $t_\ell=t_\ell(n,i,\omega):=\big(\zeta_{i,\ell}\wedge\wp_n\big)(\omega)$, $\forall\,\ell>i$. Let $\big\{t_{\widetilde{\ell}}\big\}_{\widetilde{\ell}\in\mathbb{N}}$ be the subsequence of $\{t_\ell\}_{\ell=i+1}^\infty$ such that $\varlimsup\limits_{\ell\to\infty}\mathscr{Z}(t_\ell,\omega)=\lim\limits_{\widetilde{\ell}\to\infty}\mathscr{Z}(t_{\widetilde{\ell}},\omega)$. The sequence $\big\{t_{\widetilde{\ell}}\big\}_{\widetilde{\ell}\in\mathbb{N}}$ in turn has a convergent subsequence $\big\{t_{\widetilde{\ell}'}\big\}_{\widetilde{\ell}'\in\mathbb{N}}$ with limit $\mathfrak{t}\in[0,\wp_n(\omega)]$. The continuity of path $\mathscr{Z}_\cdot(\omega)$ shows that $\mathscr{Z}(\mathfrak{t},\omega)=\lim\limits_{\widetilde{\ell}'\to\infty}\mathscr{Z}(t_{\widetilde{\ell}'},\omega)=\varlimsup\limits_{\ell\to\infty}\mathscr{Z}(t_\ell,\omega)$. Also, (6.81) implies that $\big(\gamma_i\wedge\wp_n\big)(\omega)\leq\varliminf\limits_{\ell\to\infty}\big(\zeta_{i,\ell}\wedge\wp_n\big)(\omega)=\varliminf\limits_{\ell\to\infty}t_\ell\leq\mathfrak{t}=\lim\limits_{\widetilde{\ell}'\to\infty}t_{\widetilde{\ell}'}\leq\varlimsup\limits_{\ell\to\infty}t_\ell=\varlimsup\limits_{\ell\to\infty}\big(\zeta_{i,\ell}\wedge\wp_n\big)(\omega)\leq\big(\gamma_{2i}\wedge\wp_n\big)(\omega)$. Hence

$$\inf_{t\in\mathcal{J}_{n,i}(\omega)}\mathscr{Z}(t,\omega)\leq\mathscr{Z}(\mathfrak{t},\omega)=\varlimsup_{\ell\to\infty}\mathscr{Z}\big(\zeta_{i,\ell}(\omega)\wedge\wp_n(\omega),\omega\big)\leq\sup_{t\in\mathcal{J}_{n,i}(\omega)}\mathscr{Z}(t,\omega),\tag{6.87}$$

where $\mathcal{J}_{n,i}(\omega):=\big[(\gamma_i\wedge\wp_n)(\omega),(\gamma_{2i}\wedge\wp_n)(\omega)\big]$.

An analogy to (6.82) shows that $\gamma_\sharp(\omega):=\inf\big\{t\in[0,T]\colon\mathscr{Z}_t(\omega)\leq L_t(\omega)\big\}\wedge T=\lim\limits_{i\to\infty}\uparrow\gamma_i(\omega)$. Since $\widehat{\mathscr{Y}_t}(\omega)=\mathscr{Y}_t(\omega)=L_t(\omega)$ over the interval $\big[0,\tau_0(\omega)\big)\supset\big[0,\wp_n(\omega)\big)$ by Proposition 5.1 (1), we can deduce from (5.7) that

$$\begin{aligned}\lim_{i\to\infty}\uparrow(\gamma_i\wedge\wp_n)(\omega)&=(\gamma_\sharp\wedge\wp_n)(\omega)=\inf\{t\in[0,\wp_n(\omega))\colon\mathscr{Z}_t(\omega)\leq L_t(\omega)\}\wedge\wp_n(\omega)=\inf\big\{t\in[0,\wp_n(\omega))\colon\mathscr{Z}_t(\omega)\leq\widehat{\mathscr{Y}_t}(\omega)\big\}\wedge\wp_n(\omega)\\&=\inf\big\{t\in[0,\wp_n(\omega))\colon\mathscr{Z}_t(\omega)=\widehat{\mathscr{Y}_t}(\omega)\big\}\wedge\wp_n(\omega)=(\gamma_*\wedge\wp_n)(\omega).\end{aligned}\tag{6.88}$$

It follows from the continuity of path $\mathscr{Z}_\cdot(\omega)$ that

$$\lim_{i\to\infty}\inf_{t\in\mathcal{J}_{n,i}(\omega)}\mathscr{Z}(t,\omega)=\lim_{i\to\infty}\sup_{t\in\mathcal{J}_{n,i}(\omega)}\mathscr{Z}(t,\omega)=\mathscr{Z}\big(\gamma_*(\omega)\wedge\wp_n(\omega),\omega\big).\tag{6.89*}$$

Then letting $i\to\infty$ in (6.87) yields that

$$\varlimsup_{i\to\infty}\varlimsup_{\ell\to\infty}\mathscr{Z}(\zeta_{i,\ell}(\omega)\wedge\wp_n(\omega),\omega)=\lim_{i\to\infty}\varlimsup_{\ell\to\infty}\mathscr{Z}(\zeta_{i,\ell}(\omega)\wedge\wp_n(\omega),\omega)=\mathscr{Z}(\gamma_*(\omega)\wedge\wp_n(\omega),\omega).\tag{6.90}$$

Since Proposition 5.1 (1) and Proposition 5.3 (4) imply that $\lim\limits_{n\to\infty}(\gamma_*\wedge\wp_n)(\omega)=(\gamma_*\wedge\tau_0)(\omega)=\gamma_*(\omega)$, letting $n\to\infty$ in (6.90), we see from the continuity of path $\mathscr{Z}_\cdot(\omega)$ again that

$$\varlimsup_{n\to\infty}\varlimsup_{i\to\infty}\varlimsup_{\ell\to\infty}\mathscr{Z}(\zeta_{i,\ell}(\omega)\wedge\wp_n(\omega),\omega)=\lim_{n\to\infty}\varlimsup_{i\to\infty}\varlimsup_{\ell\to\infty}\mathscr{Z}(\zeta_{i,\ell}(\omega)\wedge\wp_n(\omega),\omega)=\mathscr{Z}(\gamma_*(\omega),\omega),\quad\forall\,\omega\in\Omega.$$

Putting this back into (6.69) and using Proposition 5.3 (3) yield that $\sup\limits_{(\mathbb{P},\gamma)\in\mathcal{P}\times\mathcal{T}}\mathbb{E}_\mathbb{P}[\mathscr{Y}_{\gamma\wedge\tau_0}]=\sup\limits_{(\mathbb{P},\gamma)\in\mathcal{P}\times\mathcal{T}}\mathbb{E}_\mathbb{P}\big[\widehat{\mathscr{Y}_\gamma}\big]\leq\mathscr{Z}_0\leq\mathbb{E}_{\mathbb{P}_*}\big[\mathscr{Z}_{\gamma_*}\big]$. Since the continuity of $\mathscr{Z}$ and the right-continuity of $\widehat{\mathscr{Y}}$ imply that $\mathscr{Z}_{\gamma_*}(\omega)=\widehat{\mathscr{Y}}_{\gamma_*}(\omega)=\mathscr{Y}_{\gamma_*\wedge\tau_0}(\omega)$, $\forall\,\omega\in\Omega$, one can further deduce (1.8) and thus (1.1). $\square$

A Appendix

A.1 Technical Lemmata

Lemma A.1. *Given* $t\in[0,T]$, *let* $\tau\in\mathcal{T}^t$ *and* $(s,\omega)\in[t,T]\times\Omega^t$. *If* $\tau(\omega)\le s$, *then* $\tau(\omega\otimes_s\Omega^s)\equiv\tau(\omega)$; *if* $\tau(\omega)\ge s$ *(resp.* $>s$*), then* $\tau(\omega\otimes_s\widetilde{\omega})\ge s$ *(resp.* $>s$*),* $\forall\,\widetilde{\omega}\in\Omega^s$ *and thus* $\tau^{s,\omega}\in\mathcal{T}^s$ *by Proposition 2.1 (2).*

Proof: Let $t\in[0,T]$, $\tau\in\mathcal{T}^t$ and $(s,\omega)\in[t,T]\times\Omega^t$. When $\widehat{s}:=\tau(\omega)\le s$, since $\omega\in A:=\{\tau=\widehat{s}\}\in\mathcal{F}^t_{\widehat{s}}\subset\mathcal{F}^t_s$, Lemma 2.1 shows that $\omega\otimes_s\Omega^s\subset A$, i.e. $\tau(\omega\otimes_s\Omega^s)\equiv\widehat{s}=\tau(\omega)$.

On the other hand, when $\tau(\omega)\ge s$ (resp. $>s$), as $\omega\in A':=\{\wp\ge s\}$ (resp. $\{\wp>s\}$)$\in\mathcal{F}^t_s$, applying Lemma 2.1 again yields that $\omega\otimes_s\Omega^s\in A'$. So $\tau(\omega\otimes_s\widetilde{\omega})\ge s$ (resp. $>s$), $\forall\,\widetilde{\omega}\in\Omega^s$. $\square$

Lemma A.2. *Assume (P2). Let* $(Y,\wp)\in\mathfrak{S}$ *and* $(t,\omega)\in[0,T]\times\Omega$. *It holds for any* $\omega'\in\Omega$, $\mathbb{P}\in\mathcal{P}_t$ *and* $\gamma\in\mathcal{T}^t$ *that*

$$\mathbb{E}_\mathbb{P}\Big[\big|\widehat{Y}^{t,\omega}_\gamma-\widehat{Y}^{t,\omega'}_\gamma\big|\Big]\le\widehat{\rho}_Y\Big((1+\kappa_\wp)\|\omega-\omega'\|_{0,t}+\sup_{r\in[t_1,t_2]}\big|\omega(r)-\omega(t_1)\big|\Big)\le\widehat{\rho}_Y\Big((1+\kappa_\wp)\|\omega-\omega'\|_{0,t}+\phi^\omega_t\big(\kappa_\wp\|\omega-\omega'\|_{0,t}\big)\Big),$$

where $t_1:=\wp(\omega)\wedge\wp(\omega')\wedge t$ *and* $t_2:=\big(\wp(\omega)\vee\wp(\omega')\big)\wedge t$.

Proof: 1) Fix $\omega'\in\Omega$. We set $t_1:=\wp(\omega)\wedge\wp(\omega')\wedge t$, $t_2:=\big(\wp(\omega)\vee\wp(\omega')\big)\wedge t$ and $\delta:=(1+\kappa_\wp)\|\omega-\omega'\|_{0,t}+\sup\limits_{r\in[t_1,t_2]}\big|\omega(r)-\omega(t_1)\big|$. Fix also $\mathbb{P}\in\mathcal{P}_t$ and $\gamma\in\mathcal{T}^t$. Let $\widetilde{\omega}\in\Omega^t$. One has

$$\Big|\widehat{Y}^{t,\omega}\big(\gamma(\widetilde{\omega}),\widetilde{\omega}\big)-\widehat{Y}^{t,\omega'}\big(\gamma(\widetilde{\omega}),\widetilde{\omega}\big)\Big|=\Big|\widehat{Y}\big(\gamma(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)-\widehat{Y}\big(\gamma(\widetilde{\omega}),\omega'\otimes_t\widetilde{\omega}\big)\Big|=\Big|Y\big(s_1(\widetilde{\omega}),\omega\otimes_t\widetilde{\omega}\big)-Y\big(s_2(\widetilde{\omega}),\omega'\otimes_t\widetilde{\omega}\big)\Big|,$$

where $s_1(\widetilde{\omega}):=\gamma(\widetilde{\omega})\wedge\wp(\omega\otimes_t\widetilde{\omega})\wedge\wp(\omega'\otimes_t\widetilde{\omega})$ and $s_2(\widetilde{\omega}):=\gamma(\widetilde{\omega})\wedge\big(\wp(\omega\otimes_t\widetilde{\omega})\vee\wp(\omega'\otimes_t\widetilde{\omega})\big)$. Since (2.5) implies that

$$s_2(\widetilde{\omega})-s_1(\widetilde{\omega})\le\big|\wp(\omega\otimes_t\widetilde{\omega})-\wp(\omega'\otimes_t\widetilde{\omega})\big|\le\kappa_\wp\|\omega\otimes_t\widetilde{\omega}-\omega'\otimes_t\widetilde{\omega}\|_{0,T}=\kappa_\wp\|\omega-\omega'\|_{0,t}<\delta,\tag{A.1}$$

one can deduce from (2.2) that

$$\begin{aligned}\Big|\widehat{Y}^{t,\omega}\big(\gamma(\widetilde{\omega}),\widetilde{\omega}\big)-\widehat{Y}^{t,\omega'}\big(\gamma(\widetilde{\omega}),\widetilde{\omega}\big)\Big|&\le\rho_Y\Big(\big(s_2(\widetilde{\omega})-s_1(\widetilde{\omega})\big)+\sup_{r\in[0,T]}\Big|(\omega\otimes_t\widetilde{\omega})\big(r\wedge s_1(\widetilde{\omega})\big)-(\omega'\otimes_t\widetilde{\omega})\big(r\wedge s_2(\widetilde{\omega})\big)\Big|\Big)\\&\le\rho_Y\Big(\kappa_\wp\|\omega-\omega'\|_{0,t}+\mathcal{I}(\widetilde{\omega})+\sup_{r\in[0,T]}\Big|(\omega\otimes_t\widetilde{\omega})\big(r\wedge s_2(\widetilde{\omega})\big)-(\omega'\otimes_t\widetilde{\omega})\big(r\wedge s_2(\widetilde{\omega})\big)\Big|\Big)\le\rho_Y\Big((1+\kappa_\wp)\|\omega-\omega'\|_{0,t}+\mathcal{I}(\widetilde{\omega})\Big),\end{aligned}\tag{A.2}$$

where $\mathcal{I}(\widetilde{\omega}):=\sup\limits_{r\in[0,T]}\big|(\omega\otimes_t\widetilde{\omega})\big(r\wedge s_1(\widetilde{\omega})\big)-(\omega\otimes_t\widetilde{\omega})\big(r\wedge s_2(\widetilde{\omega})\big)\big|=\sup\limits_{r\in[s_1(\widetilde{\omega}),s_2(\widetilde{\omega})]}\big|(\omega\otimes_t\widetilde{\omega})(r)-(\omega\otimes_t\widetilde{\omega})\big(s_1(\widetilde{\omega})\big)\big|$.

2) Next, we discuss by three cases on values of $\wp(\omega)$ and $\wp(\omega')$:

(i) When $\wp(\omega)\wedge\wp(\omega')\ge t$, Lemma A.1 shows that $\wp^{t,\omega}$ and $\wp^{t,\omega'}$ belong to $\mathcal{T}^t$, so does $\zeta:=\gamma\wedge\wp^{t,\omega}\wedge\wp^{t,\omega'}$. For any $\widetilde{\omega}\in\Omega^t$, as $s_1(\widetilde{\omega})=\zeta(\widetilde{\omega})\ge t$, (A.1) implies that $\mathcal{I}(\omega)=\sup\limits_{r\in[s_1(\widetilde{\omega}),s_2(\widetilde{\omega})]}\big|\widetilde{\omega}(r)-\widetilde{\omega}(s_1(\widetilde{\omega}))\big|\le\sup\limits_{r\in[\zeta(\widetilde{\omega}),(\zeta(\widetilde{\omega})+\delta)\wedge T]}\big|B^t_r(\widetilde{\omega})-B^t_\zeta(\widetilde{\omega})\big|$. Putting it back into (A.2) and taking expectation $\mathbb{E}_\mathbb{P}[\ ]$, we see from (3.5) that

$$\mathbb{E}_\mathbb{P}\Big[\big|\widehat{Y}^{t,\omega}_\gamma-\widehat{Y}^{t,\omega'}_\gamma\big|\Big]\le\mathbb{E}_\mathbb{P}\Big[\rho_Y\Big(\delta+\sup_{r\in[\zeta,(\zeta+\delta)\wedge T]}\big|B^t_r-B^t_\zeta\big|\Big)\Big]\le\widehat{\rho}_Y(\delta).\tag{A.3}$$

(ii) When $\wp(\omega)\wedge\wp(\omega')<t\le\wp(\omega)\vee\wp(\omega')$, let $(\underline{\omega},\overline{\omega})$ be a possible permutation of (ω,ω') such that $\wp(\underline{\omega})=\wp(\omega)\wedge\wp(\omega')<t$ and $\wp(\overline{\omega})=\wp(\omega)\vee\wp(\omega')\ge t$. By Lemma A.1, $\wp(\underline{\omega}\otimes_t\Omega^t)\equiv\wp(\underline{\omega})$ and $\wp(\overline{\omega}\otimes_t\Omega^t)\subset[t,T]$. For any $\widetilde{\omega}\in\Omega^t$, one has $s_1(\widetilde{\omega})=\gamma(\widetilde{\omega})\wedge\wp(\underline{\omega}\otimes_t\widetilde{\omega})\wedge\wp(\overline{\omega}\otimes_t\widetilde{\omega})=\wp(\underline{\omega})=t_1<t$ and $s_2(\widetilde{\omega})=\gamma(\widetilde{\omega})\wedge\big(\wp(\underline{\omega}\otimes_t\widetilde{\omega})\vee\wp(\overline{\omega}\otimes_t\widetilde{\omega})\big)=\gamma(\widetilde{\omega})\wedge\wp(\overline{\omega}\otimes_t\widetilde{\omega})\ge t$. Since $s_2(\widetilde{\omega})<s_1(\widetilde{\omega})+\delta<t+\delta$ by (A.1) and since $t_2=\wp(\overline{\omega})\wedge t=t$, we can deduce that

$$\begin{aligned}\mathcal{I}(\widetilde{\omega})&=\Big(\sup_{r\in[s_1(\widetilde{\omega}),t]}\big|\omega(r)-\omega\big(s_1(\widetilde{\omega})\big)\big|\Big)\vee\Big(\sup_{r\in[t,s_2(\widetilde{\omega})]}\big|\widetilde{\omega}(r)+\omega(t)-\omega\big(s_1(\widetilde{\omega})\big)\big|\Big)\\&\le\Big(\sup_{r\in[t_1,t]}\big|\omega(r)-\omega(t_1)\big|\Big)\vee\Big(\big|\omega(t)-\omega(t_1)\big|+\sup_{r\in[t,s_2(\widetilde{\omega})]}\big|\widetilde{\omega}(r)-\widetilde{\omega}(t)\big|\Big)\le\sup_{r\in[t_1,t_2]}\big|\omega(r)-\omega(t_1)\big|+\sup_{r\in[t,(t+\delta)\wedge T]}\big|B^t_r(\widetilde{\omega})-B^t_t(\widetilde{\omega})\big|.\end{aligned}$$

An analogy to (A.3) shows that

$$\mathbb{E}_\mathbb{P}\Big[\big|\widehat{Y}^{t,\omega}_\gamma-\widehat{Y}^{t,\omega'}_\gamma\big|\Big]\le\mathbb{E}_\mathbb{P}\Big[\rho_Y\Big(\delta+\sup_{r\in[t,(t+\delta)\wedge T]}\big|B^t_r-B^t_t\big|\Big)\Big]\le\widehat{\rho}_Y(\delta).\tag{A.4}$$

(iii) When $\wp(\omega)\vee\wp(\omega')<t$, we see from Lemma A.1 again that $\wp(\omega\otimes_t\Omega^t)\equiv\wp(\omega)<t$ and $\wp(\omega'\otimes_t\Omega^t)\equiv\wp(\omega')<t$. For any $\widetilde{\omega}\in\Omega^t$, as $\gamma(\widetilde{\omega})\ge t$, one has $s_1(\widetilde{\omega})=\wp(\omega)\wedge\wp(\omega')=t_1<t$ and $s_2(\widetilde{\omega})=\wp(\omega)\vee\wp(\omega')=t_2<t$. It follows that $\mathcal{I}(\omega)=\sup\limits_{r\in[t_1,t_2]}\big|\omega(r)-\omega(t_1)\big|$, then (A.4) still holds for this case.

Therefore, we have proved the first inequality of the lemma. Since $t_2-t_1=\big|\wp(\omega)\wedge t-\wp(\omega')\wedge t\big|\le|\wp(\omega)-\wp(\omega')|\le\kappa_\wp\|\omega-\omega'\|_{0,t}$ by (2.5), the second inequality easily follows. □

Lemma A.3. *Assume (P2)–(P4) and let $(Y,\wp)\in\mathfrak{S}$. Given $\mathbb{P}\in\mathcal{P}$, Z is a $\mathbb{P}-$supermartingale and $\mathbb{E}_\mathbb{P}[Z_\tau]\ge\mathbb{E}_\mathbb{P}[Z_\gamma]$ holds for any $\tau,\gamma\in\mathcal{T}$ with $\tau\le\gamma$, $\mathbb{P}-$a.s.*

Proof: Fix $(Y,\wp)\in\mathfrak{S}$ and $\mathbb{P}\in\mathcal{P}$.
1) Let $t\in[0,T]$ and $\gamma\in\mathcal{T}$. Proposition 4.1 and (4.1) show that Z_γ is an $\mathcal{F}_T-$measurable bounded random variable. By Proposition 2.2, we can find a $\mathbb{P}-$null set $\mathcal{N}$ such that $\mathbb{E}_\mathbb{P}[Z_\gamma|\mathcal{F}_t](\omega)=\mathbb{E}_{\mathbb{P}^{t,\omega}}\big[(Z_\gamma)^{t,\omega}\big]$, $\forall\,\omega\in\mathcal{N}^c$. Also, (P3) shows that for some extension $(\Omega,\mathcal{F}',\mathbb{P}')$ of $(\Omega,\mathcal{F}_T,\mathbb{P})$ and some $\Omega'\in\mathcal{F}'$ with $\mathbb{P}'(\Omega')=1$, $\mathbb{P}^{t,\omega}\in\mathcal{P}_t$ for any $\omega\in\Omega'$. Then Proposition 4.3 implies that $\mathbb{E}_\mathbb{P}[Z_\gamma|\mathcal{F}_t](\omega)=\mathbb{E}_{\mathbb{P}^{t,\omega}}\big[(Z_\gamma)^{t,\omega}\big]\le\overline{\mathscr{E}}_t\big[Z_\gamma\big](\omega)\le Z_{\gamma\wedge t}(\omega)$, $\forall\,\omega\in\Omega'\cap\mathcal{N}^c$. Using similar arguments that lead to (6.5), we can obtain that

$$\mathbb{E}_\mathbb{P}[Z_\gamma|\mathcal{F}_t]\le Z_{\gamma\wedge t},\quad \mathbb{P}-\text{a.s.} \tag{A.5}$$

2) Let $\tau,\gamma\in\mathcal{T}$ with $\tau\le\gamma$, $\mathbb{P}-$a.s. Also, let $n\in\mathbb{N}$ and $i=1,\cdots,2^n$. We set $t_i^n:=i2^{-n}T$ and $A_i^n:=\{t_{i-1}^n<\tau\le t_i^n\}\in\mathcal{F}_{t_i^n}$ with $t_0^n:=0$. Applying (A.5) with $t=t_i^n$ yields that $\mathbb{E}_\mathbb{P}[Z_\gamma|\mathcal{F}_{t_i^n}]\le Z_{\gamma\wedge t_i^n}$, $\mathbb{P}-$a.s. Multiplying $\mathbf{1}_{A_i^n}$ and taking summation over $i\in\{1,\cdots,2^n\}$, we obtain $\mathbb{E}_\mathbb{P}[Z_\gamma|\mathcal{F}_{\tau_n}]\le Z_{\gamma\wedge\tau_n}$, $\mathbb{P}-$a.s., where $\tau_n:=\sum\limits_{i=1}^{2^n}\mathbf{1}_{A_i^n}t_i^n\in\mathcal{T}$. Then taking the expectation $\mathbb{E}_\mathbb{P}[\;]$ yields that $\mathbb{E}_\mathbb{P}[Z_\gamma]\le\mathbb{E}_\mathbb{P}[Z_{\gamma\wedge\tau_n}]$. Since $\lim\limits_{n\to\infty}\downarrow\tau_n=\tau$ and since Proposition 4.2 shows that Z is a bounded process with all continuous paths, an application of the bounded convergence theorem leads to that $\mathbb{E}_\mathbb{P}[Z_\gamma]\le\mathbb{E}_\mathbb{P}[Z_{\gamma\wedge\tau}]=\mathbb{E}_\mathbb{P}[Z_\tau]$. □

We need the following extension of Lemma 4.5 of [19] to prove Theorem 4.1 and Theorem 3.1.

Lemma A.4. *Assume (P1). Let $\Omega_0\subset\Omega$ and let $\underline{\theta}$, θ, $\overline{\theta}$ be three real-valued random variables on Ω taking values in a compact interval $I\subset\mathbb{R}$ with length $|I|>0$. If for any $\omega\in\Omega_0$ there exists a $\delta(\omega)>0$ such that*

$$\underline{\theta}(\omega')\le\theta(\omega)\le\overline{\theta}(\omega'),\quad \forall\,\omega'\in\overline{O}_{\delta(\omega)}(\omega)=\big\{\omega'\in\Omega:\|\omega'-\omega\|_{0,T}\le\delta(\omega)\big\}, \tag{A.6}$$

then for any $\varepsilon>0$ one can find an open subset $\widehat{\Omega}$ of Ω and a Lipschitz continuous random variable $\widehat{\theta}:\Omega\to I$ such that $\sup\limits_{\mathbb{P}\in\mathcal{P}}\mathbb{P}\big(\widehat{\Omega}^c\big)\le\varepsilon$ and that $\underline{\theta}-\varepsilon<\widehat{\theta}<\overline{\theta}+\varepsilon$ on $\widehat{\Omega}\cap\Omega_0$.

Proof: Since the canonical space Ω is a separable complete metric space and thus Lindelöf, there exists a sequence $\{\omega_j\}_{j\in\mathbb{N}}$ of Ω such that $\underset{j\in\mathbb{N}}{\cup}O_j=\Omega$ with $O_j:=O_{\frac12\delta(\omega_j)}(\omega_j)=\{\omega\in\Omega:\|\omega-\omega_j\|_{0,T}<\frac12\delta(\omega_j)\}$.

Let $n\in\mathbb{N}$ with $n^2>|I|^{-1}$. By (2.1), $\Omega_n:=\overset{n}{\underset{j=1}{\cup}}O_j$ is an open subset of Ω. For $j=1,\cdots,n$, we define function $f_{n,j}:[0,\infty)\to[0,1]$ by: $f_{n,j}(x):=1$ for $x\in\big[0,\frac12\delta(\omega_j)\big]$, $f_{n,j}(x):=n^{-2}|I|^{-1}$ for $x\ge\delta(\omega_j)$, and $f_{n,j}$ is linear in $\big[\frac12\delta(\omega_j),\delta(\omega_j)\big]$. Clearly, $g_{n,j}(\omega):=f_{n,j}(\|\omega-\omega_j\|_{0,T})$, $\omega\in\Omega$ is a Lipschitz continuous random variable on Ω with coefficient $<2/\delta(\omega_j)$. It follows that $\mathfrak{g}_n:=\sum_{j=1}^n g_{n,j}$ is a Lipschitz continuous random variable on Ω with values in $\big[n^{-1}|I|^{-1},n\big]$ and that $\sum_{j=1}^n\theta(\omega_j)g_{n,j}$ is a Lipschitz continuous random variable on Ω whose absolute values $\le\sum_{j=1}^n|\theta(\omega_j)|$. Then one can deduce that

$$\theta_n(\omega):=\frac{1}{\mathfrak{g}_n(\omega)}\sum_{j=1}^n\theta(\omega_j)g_{n,j}(\omega),\quad \forall\,\omega\in\Omega$$

defines another Lipschitz continuous random variable on Ω with values in I.

Given $\omega\in\Omega_n\cap\Omega_0$, as ω belongs O_j for some $j=1,\cdots,n$, we see that the index set $J_n(\omega):=\{1\le j\le n: \|\omega-\omega_j\|_{0,T}\le\delta(\omega_j)\}$ is not empty and that $\mathfrak{g}_n(\omega)>1$. Then one can deduce from (A.6) that

$$\begin{aligned}\theta_n(\omega)-\overline{\theta}(\omega) &= \frac{1}{\mathfrak{g}_n(\omega)}\left(\sum_{j\in J_n(\omega)}[\theta(\omega_j)-\overline{\theta}(\omega)]g_{n,j}(\omega)+\sum_{j\notin J_n(\omega)}[\theta(\omega_j)-\overline{\theta}(\omega)]g_{n,j}(\omega)\right)\\ &\le \frac{1}{\mathfrak{g}_n(\omega)}\sum_{j\notin J_n(\omega)}|I|g_{n,j}(\omega)=\frac{1}{\mathfrak{g}_n(\omega)}\sum_{j\notin J_n(\omega)}\frac{1}{n^2}<\frac{1}{n},\end{aligned}$$

and similarly, $\theta_n(\omega)-\underline{\theta}(\omega)>-\frac{1}{n}$. Since $\mathcal{P}$ is a weakly compact subset of $\mathfrak{P}_0$ by (P1) and since $\underset{n\in\mathbb{N}}{\cup}\Omega_n=\Omega$, Lemma 8 of [16] shows that $\lim_{n\to\infty}\downarrow\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{P}\big(\Omega_n^c\big)=0$. Hence, for any $\varepsilon>0$, there exists an integer $N>1/\varepsilon$ such that for any $n\ge N$, $\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{P}\big(\Omega_n^c\big)\le\varepsilon$. Then we take $\big(\widehat{\Omega},\widehat{\theta}\big)=\big(\Omega_N,\theta_N\big)$. □

One can find $\mathbf{F}$−stopping times that are locally Lipschitz continuous as follows. This result and its consequence, Lemma A.6, are crucial for our approximating τ_0 by Lipschitz continuous stopping times in Proposition 5.1.

Lemma A.5. *Let $(T_0,\omega_0)\in(0,T]\times\Omega$ and $\mathfrak{R},\kappa>0$. There exists an $\mathbf{F}$−stopping time ζ valued in $(0,T_0]$ such that $\zeta\equiv T_0$ on $\overline{O}^{T_0}_{\mathfrak{R}}(\omega_0)=\{\omega\in\Omega:\|\omega-\omega_0\|_{0,T_0}\le\mathfrak{R}\}$ and that given $\omega_1,\omega_2\in\Omega$,*

$$|\zeta(\omega_1)-\zeta(\omega_2)|\le\kappa\|\omega_1-\omega_2\|_{0,t_0} \tag{A.7}$$

holds for any $t_0\in[b,T_0]\cup\big\{t\in[a,b): t\ge a+\kappa\|\omega_1-\omega_2\|_{0,t}\big\}$, where $a:=\zeta(\omega_1)\wedge\zeta(\omega_2)$ and $b:=\zeta(\omega_1)\vee\zeta(\omega_2)$.

Proof: Given $(t,\omega)\in[0,T]\times\Omega$, the continuity of paths $\omega(\cdot)$, $\omega_0(\cdot)$ implies that

$$X_t(\omega):=\|\omega-\omega_0\|_{0,t}=\sup_{r\in[0,t]}|B_r(\omega)-\omega_0(r)|=\sup_{r\in\mathbb{Q}\cap[0,t]}|B_r(\omega)-\omega_0(r)|\in[0,\infty).$$

As the random variable $\sup_{r\in\mathbb{Q}\cap[0,t]}|B_r-\omega_0(r)|$ is $\mathcal{F}_t$−measurable, we see that X is an $\mathbf{F}$−adapted process with all continuous paths.

Define $f(x):=-x/\kappa+T_0/\kappa+\mathfrak{R}$, $\forall\,x\in[0,T_0]$. Since $\zeta_0:=\inf\{t\in[0,T]: f(t\wedge T_0)-X_t\le 0\}\wedge T$ is an $\mathbf{F}$−stopping time, $\zeta:=\zeta_0\wedge T_0=\inf\{t\in[0,T_0]: X_t\ge f(t)\}\wedge T_0$ is also an $\mathbf{F}$−stopping time taking values in $(0,T_0]$: Given $\omega\in\Omega$, since $X_0(\omega)-f(0)=0-(T_0/\kappa+\mathfrak{R})<0$ and since the path $X_\cdot(\omega)-f(\cdot)$ is continuous, there exists some $t_\omega\in(0,T_0)$ such that $X_t(\omega)-f(t)\le-\frac{1}{2}(T_0/\kappa+\mathfrak{R})<0$, $\forall\,t\in[0,t_\omega]$. Thus $\zeta(\omega)>t_\omega>0$.

Let $\omega\in\Omega$. If $\|\omega-\omega_0\|_{0,T_0}\le\mathfrak{R}$, one can deduce that $X_t(\omega)=\|\omega-\omega_0\|_{0,t}\le\|\omega-\omega_0\|_{0,T_0}\le\mathfrak{R}=f(T_0)<f(t)$, $\forall\,t\in[0,T_0)$, thus, $\zeta(\omega)=T_0$.

Next, let $\omega_1,\omega_2\in\Omega$. If $\zeta(\omega_1)=\zeta(\omega_2)$, (A.7) holds automatically. So let us assume without loss of generality that $a:=\zeta(\omega_1)<\zeta(\omega_2):=b$. We claim that

$$\text{if } t_0\in[a,b] \text{ satisfies } t_0-a\ge\kappa\|\omega_1-\omega_2\|_{0,t_0}, \text{ then } |\zeta(\omega_1)-\zeta(\omega_2)|=b-a\le\kappa\|\omega_1-\omega_2\|_{0,t_0}. \tag{A.8}$$

To see this, we let $t_0\in[a,b]$ satisfying $t_0-a\ge\kappa\|\omega_1-\omega_2\|_{0,t_0}$, and set $\delta:=\|\omega_1-\omega_2\|_{0,t_0}$, $\widehat{t}:=a+\kappa\delta\le t_0$. As $\zeta(\omega_1)<T_0$, the continuity of process X and function f implies that $\|\omega_1-\omega_0\|_{0,a}=\|\omega_1-\omega_0\|_{0,\zeta(\omega_1)}=f(\zeta(\omega_1))=f(a)$. Then one can deduce that

$$\|\omega_2-\omega_0\|_{0,\widehat{t}}\ge\|\omega_1-\omega_0\|_{0,\widehat{t}}-\|\omega_1-\omega_2\|_{0,\widehat{t}}\ge\|\omega_1-\omega_0\|_{0,a}-\|\omega_1-\omega_2\|_{0,t_0}=f(a)-\delta=f\big(\widehat{t}\big).$$

So $b=\zeta(\omega_2)\le\widehat{t}$. It follows that $|\zeta(\omega_1)-\zeta(\omega_2)|=b-a\le\widehat{t}-a=\kappa\delta=\kappa\|\omega_1-\omega_2\|_{0,t_0}$, proving the claim.

If $b-a>\kappa\|\omega_1-\omega_2\|_{0,b}$ held, applying (A.8) with $t_0=b$ would yield that $b-a=|\zeta(\omega_1)-\zeta(\omega_2)|\le\kappa\|\omega_1-\omega_2\|_{0,b}$, a contradiction appears. Hence, we must have $|\zeta(\omega_1)-\zeta(\omega_2)|=b-a\le\kappa\|\omega_1-\omega_2\|_{0,b}\le\kappa\|\omega_1-\omega_2\|_{0,t_0}$, $\forall\,t_0\in[b,T_0]$. □

Lemma A.6. *Let θ_1, θ_2, θ_3 be three real-valued random variables on Ω satisfying: for some $\delta>0$, it holds for $i=1,2$ and any $\omega\in\Omega$ that*

$$\theta_i(\omega')\le\theta_{i+1}(\omega),\qquad\forall\,\omega'\in\overline{O}^{\theta_{i+1}(\omega)}_\delta(\omega)=\big\{\omega'\in\Omega:\|\omega'-\omega\|_{0,\theta_{i+1}(\omega)}\le\delta\big\}. \tag{A.9}$$

If θ_2 takes values in $(0,T]$, then for any $\kappa > T/\delta$, there exists an $\mathbf{F}-$stopping time $\wp$ such that $\theta_1 \le \wp \le \theta_3$ on Ω. Moreover, given $\omega_1, \omega_2 \in \Omega$,

$$\big|\wp(\omega_1) - \wp(\omega_2)\big| \le \kappa \|\omega_1 - \omega_2\|_{0,t_0} \tag{A.10}$$

holds for any $t_0 \in [b,T] \cup \big\{t \in [a,b) \colon t \ge a + \kappa\|\omega_1 - \omega_2\|_{0,t}\big\}$, where $a := \wp(\omega_1) \wedge \wp(\omega_2)$ and $b := \wp(\omega_1) \vee \wp(\omega_2)$.

Proof: We fix $\kappa > T/\delta$ and set $\delta_0 := \delta - T/\kappa$. Since the canonical space Ω is a separable complete metric space and thus Lindelöf, there exists a countable dense subset $\big\{\omega_j\big\}_{j\in\mathbb{N}}$ of Ω under norm $\|\ \|_{0,T}$. Given $j \in \mathbb{N}$, we set $t_j := \theta_2(\omega_j) \in (0,T]$ and $\kappa_j := \frac{t_j}{\delta - \delta_0}$. Applying Lemma A.5 with $(\omega_0, T_0, \mathfrak{R}, \kappa) = (\omega_j, t_j, \delta_0, \kappa_j)$ yields an $\mathbf{F}-$stopping time ζ_j valued in $(0, t_j]$ such that

$$\zeta_j(\omega) \equiv t_j, \quad \forall\, \omega \in \overline{O}^{t_j}_{\delta_0}(\omega_j). \tag{A.11}$$

Given $\omega_1, \omega_2 \in \Omega$, it holds for any $t_0 \in [b_j, t_j] \cup \big\{t \in [a_j, b_j) \colon t \ge a_j + \kappa_j \|\omega_1 - \omega_2\|_{0,t}\big\}$ that

$$\big|\zeta_j(\omega_1) - \zeta_j(\omega_2)\big| \le \kappa_j \|\omega_1 - \omega_2\|_{0,t_0} \le \kappa \|\omega_1 - \omega_2\|_{0,t_0}, \tag{A.12}$$

where $a_j := \zeta_j(\omega_1) \wedge \zeta_j(\omega_2)$ and $b_j := \zeta_j(\omega_1) \vee \zeta_j(\omega_2)$.

Clearly, $\wp := \sup\limits_{j\in\mathbb{N}} \zeta_j$ defines an $\mathbf{F}-$stopping time taking values in $(0,T]$. Let $\omega_1, \omega_2 \in \Omega$. If $\wp(\omega_1) = \wp(\omega_2)$, one has (A.10) automatically. So let us assume without loss of generality that $a := \wp(\omega_1) < \wp(\omega_2) := b$. We claim that

$$\text{if } t_0 \in [a,b] \text{ satisfies } t_0 - a \ge \kappa \|\omega_1 - \omega_2\|_{0,t_0}, \text{ then } \big|\wp(\omega_1) - \wp(\omega_2)\big| \le \kappa \|\omega_1 - \omega_2\|_{0,t_0}. \tag{A.13}$$

To see this, we let $t_0 \in [a,b]$ satisfying $t_0 - a \ge \kappa \|\omega_1 - \omega_2\|_{0,t_0}$, and let $\lambda \in (0, b-a]$. There exists a $j = j(\lambda) \in \mathbb{N}$ such that $\zeta_j(\omega_2) \ge b - \lambda$. As $\zeta_j(\omega_2) \ge a = \wp(\omega_1) \ge \zeta_j(\omega_1)$, we see that $a_j = \zeta_j(\omega_1)$ and $b_j = \zeta_j(\omega_2)$. Then t_0 is in $[a_j, T]$ and satisfies $t_0 - a_j \ge t_0 - a \ge \kappa \|\omega_1 - \omega_2\|_{0,t_0} \ge \kappa_j \|\omega_1 - \omega_2\|_{0,t_0}$. So by (A.12), $\big|\wp(\omega_1) - \wp(\omega_2)\big| = b - a \le \zeta_j(\omega_2) + \lambda - \zeta_j(\omega_1) \le \kappa \|\omega_1 - \omega_2\|_{0,t_0} + \lambda$. Letting $\lambda \to 0$ yields that $\big|\wp(\omega_1) - \wp(\omega_2)\big| \le \kappa \|\omega_1 - \omega_2\|_{0,t_0}$, proving the claim.

If $b - a > \kappa \|\omega_1 - \omega_2\|_{0,b}$ held, applying claim (A.13) with $t_0 = b$ would yield that $b - a = |\wp(\omega_1) - \wp(\omega_2)| \le \kappa \|\omega_1 - \omega_2\|_{0,b}$, a contradiction appears. Hence, we must have $|\wp(\omega_1) - \wp(\omega_2)| = b - a \le \kappa \|\omega_1 - \omega_2\|_{0,b} \le \kappa \|\omega_1 - \omega_2\|_{0,t_0}$, $\forall\, t_0 \in [b,T]$.

Now, let us fix $\omega \in \Omega$. Since $O_{\delta_0}\big(\omega_j\big) \subset O^{t_j}_{\delta_0}\big(\omega_j\big)$ for any $j \in \mathbb{N}$, one has $\Omega = \underset{j\in\mathbb{N}}{\cup} O_{\delta_0}\big(\omega_j\big) \subset \underset{j\in\mathbb{N}}{\cup} O^{t_j}_{\delta_0}\big(\omega_j\big) \subset \Omega$. So $\omega \in O^{t_j}_{\delta_0}\big(\omega_j\big)$ for some $j \in \mathbb{N}$ and it follows from (A.11) that $\wp(\omega) \ge \zeta_j(\omega) = t_j > 0$. Since $\|\omega - \omega_j\|_{0,\theta_2(\omega_j)} = \|\omega - \omega_j\|_{0,t_j} < \delta_0 < \delta$, taking $(i, \omega, \omega') = (1, \omega_j, \omega)$ in (A.9) shows that $\theta_1(\omega) \le \theta_2(\omega_j) = t_j = \zeta_j(\omega) \le \wp(\omega)$.

We claim that $\zeta_\ell(\omega) \le \theta_3(\omega)$, $\forall\, \ell \in \mathbb{N}$: Assume not, i.e. $\zeta_\ell(\omega) > \theta_3(\omega)$ for some $\ell \in \mathbb{N}$. From the proof of Lemma A.5, we see that $\zeta_\ell(\omega) = \inf\{t \in [0, t_\ell] \colon \|\omega - \omega_\ell\|_{0,t} \ge f_\ell(t)\} \wedge t_\ell$, where $f_\ell(x) := -x/\kappa_\ell + t_j/\kappa_\ell + \delta_0$, $\forall\, x \in [0, t_\ell]$. Since $\|\omega_\ell - \omega\|_{0,\theta_3(\omega)} \le \|\omega - \omega_\ell\|_{0,\zeta_\ell(\omega)} \le f_\ell\big(\zeta_\ell(\omega)\big) < f_\ell(0) = t_\ell/\kappa_\ell + \delta_0 = \delta$, taking $(i, \omega, \omega') = \big(2, \omega, \omega_\ell\big)$ in (A.9) leads to a contradiction: $\theta_3(\omega) \ge \theta_2\big(\omega_\ell\big) = t_\ell \ge \zeta_\ell(\omega)$! Hence, $\zeta_\ell(\omega) \le \theta_3(\omega)$, $\forall\, \ell \in \mathbb{N}$. It follows that $\wp(\omega) = \sup\limits_{\ell\in\mathbb{N}} \zeta_\ell(\omega) \le \theta_3(\omega)$. □

A.2 Proofs of Starred Inequalities in Section 6

Proof of (6.12)**:** Let $r \in [t,T]$. If $r < t_1$, as $\{\gamma < \nu\} \in \mathcal{F}^t_{\gamma\wedge\nu} \subset \mathcal{F}^t_\gamma$, one has $\{\widehat{\gamma}_\lambda \le r\} = \{\gamma < \nu\} \cap \{\gamma \le r\} \in \mathcal{F}^t_r$. Otherwise, if $r \ge t_1$, let k be the largest integer such that $t_k \le r$. Since $\{\gamma \ge \nu\} \cap \{\gamma \le r\} \subset \{\nu \le r\} \subset \{\nu \ne t_i\} \subset \mathcal{A}^i_0$ for $i = k+1 \cdots, m$ and since $\{\gamma \ge \nu\} \cap \mathcal{A}^i_j = \{\gamma \ge t_i\} \cap \{\nu = t_i\} \cap \Big(O^{t_i}_{\delta_j}(\widetilde{\omega}_j) \backslash \underset{j'<j}{\cup} O^{t_i}_{\delta_{j'}}(\widetilde{\omega}_{j'})\Big) \in \mathcal{F}^t_{t_i} \subset \mathcal{F}^t_r$ for $i = 1, \cdots, k$ and $j = 1, \cdots, \lambda$, one can deduce that

$$\{\widehat{\gamma}_\lambda \le r\} = \big(\{\gamma < \nu\} \cap \{\gamma \le r\}\big) \cup \bigg[\{\gamma \ge \nu\} \cap \{\gamma \le r\} \cap \Big(\bigcap_{i=1}^{k} \mathcal{A}^i_0\Big)\bigg] \cup \bigg[\bigcup_{i=1}^{k} \bigcup_{j=1}^{\lambda} \Big(\{\gamma \ge \nu\} \cap \mathcal{A}^i_j \cap \{\gamma^i_j(\Pi^t_{t_i}) \le r\}\Big)\bigg] \in \mathcal{F}^t_r.$$

Hence, $\widehat{\gamma}_\lambda \in \mathcal{T}^t$.

Proof of (6.41)**:** We let $\widehat{\kappa}_n$ be the Lipschitz coefficient of $\widehat{\delta}_n$. Given $\omega \in \Omega$ and $\varepsilon > 0$, set $\widehat{\lambda}_n = \widehat{\lambda}_n(\omega, \varepsilon) := \frac{\varepsilon}{3} \wedge \frac{(\phi^\omega_T)^{-1}(\varepsilon/3)}{\widehat{\kappa}_n}$ and let $\omega' \in O_{\widehat{\lambda}_n}(\omega)$.

Let $0 \le r \le r' \le T$ with $r' - r \le \widehat{\delta}_n(\omega)$. If $\widehat{\delta}_n(\omega) \le \widehat{\delta}_n(\omega')$, then

$$|\omega(r') - \omega(r)| \le |\omega(r') - \omega'(r')| + |\omega'(r') - \omega'(r)| + |\omega'(r) - \omega(r)| \le \phi^{\omega'}_T\big(\widehat{\delta}_n(\omega)\big) + 2\|\omega' - \omega\|_{0,T} < \phi^{\omega'}_T\big(\widehat{\delta}_n(\omega')\big) + \frac{2}{3}\varepsilon. \tag{A.14}$$

Otherwise if $\widehat{\delta}_n(\omega')<\widehat{\delta}_n(\omega)$, we set $s':=r'\wedge(r+\widehat{\delta}_n(\omega'))$. Since (2.5) shows that $r'-s'=r'\wedge(r+\widehat{\delta}_n(\omega))-r'\wedge(r+\widehat{\delta}_n(\omega'))\le \widehat{\delta}_n(\omega)-\widehat{\delta}_n(\omega')\le\widehat{\kappa}_n\|\omega-\omega'\|_{0,T}$ and that $s'-r=r'\wedge(r+\widehat{\delta}_n(\omega'))-r'\wedge r\le\widehat{\delta}_n(\omega')$, we can deduce that

$$\begin{aligned}|\omega(r')-\omega(r)| &\le |\omega(r')-\omega(s')|+|\omega(s')-\omega(r)|\le\phi^\omega_T\big(\widehat{\kappa}_n\|\omega-\omega'\|_{0,T}\big)+|\omega(s')-\omega'(s')|+|\omega'(s')-\omega'(r)|+|\omega'(r)-\omega(r)|\\ &\le \phi^\omega_T\big(\widehat{\kappa}_n\|\omega-\omega'\|_{0,T}\big)+\phi^{\omega'}_T\big(\widehat{\delta}_n(\omega')\big)+2\|\omega'-\omega\|_{0,T}<\phi^{\omega'}_T\big(\widehat{\delta}_n(\omega')\big)+\varepsilon.\end{aligned}$$

Combining it with (A.14) and taking supremum over the pair (r,r') yields that $\phi^\omega_T\big(\widehat{\delta}_n(\omega)\big)\le\phi^{\omega'}_T\big(\widehat{\delta}_n(\omega')\big)+\varepsilon$.

On the other hand, let $0\le\widetilde{r}\le\widetilde{r}'\le T$ with $\widetilde{r}'-\widetilde{r}\le\widehat{\delta}_n(\omega')$. If $\widehat{\delta}_n(\omega')\le\widehat{\delta}_n(\omega)$, an analogy to (A.14) shows that

$$|\omega'(\widetilde{r}')-\omega'(\widetilde{r})|<\phi^\omega_T\big(\widehat{\delta}_n(\omega)\big)+\frac{2}{3}\varepsilon. \tag{A.15}$$

Otherwise if $\widehat{\delta}_n(\omega)<\widehat{\delta}_n(\omega')$, one can deduce that

$$\begin{aligned}&|\omega'(\widetilde{r}')-\omega'(\widetilde{r})|\le|\omega'(\widetilde{r}')-\omega(\widetilde{r}')|+|\omega(\widetilde{r}')-\omega(\widetilde{r})|+|\omega(\widetilde{r})-\omega'(\widetilde{r})|\le\phi^\omega_T\big(\widehat{\delta}_n(\omega')\big)+2\|\omega-\omega'\|_{0,T}\\ &\quad\le\phi^\omega_T\big(\widehat{\delta}_n(\omega')-\widehat{\delta}_n(\omega)\big)+\phi^\omega_T\big(\widehat{\delta}_n(\omega)\big)+2\|\omega-\omega'\|_{0,T}<\phi^\omega_T\big(\widehat{\kappa}_n\|\omega-\omega'\|_{0,T}\big)+\phi^\omega_T\big(\widehat{\delta}_n(\omega)\big)+\frac{2}{3}\varepsilon\le\phi^\omega_T\big(\widehat{\delta}_n(\omega)\big)+\varepsilon.\end{aligned}$$

Combining it with (A.15) and taking supremum over the pair $\big(\widetilde{r},\widetilde{r}'\big)$ yields that $\phi^{\omega'}_T\big(\widehat{\delta}_n(\omega')\big)\le\phi^\omega_T\big(\widehat{\delta}_n(\omega)\big)+\varepsilon$.

Hence $\omega\to\phi^\omega_T\big(\widehat{\delta}_n(\omega)\big)$ is a continuous random variable on Ω. □

Proof of (6.42)**:** Let $\omega,\omega'\in\Omega$ and set $t:=\widehat{\theta}_n(\omega)$, $s:=\widehat{\theta}_n(\omega')$. We see from (4.2) and (4.5) that

$$\begin{aligned}&\big|Z_s(\omega)-Z_s(\omega')\big|\le\widehat{\rho}_Y\Big((1+\kappa_\wp)\|\omega-\omega'\|_{0,T}+\phi^\omega_T\big(\kappa_\wp\|\omega-\omega'\|_{0,T}\big)\Big),\quad\text{and}\\ &\big|Z_t(\omega)-Z_s(\omega)\big|=\big|Z_{t\wedge s}(\omega)-Z_{t\vee s}(\omega)\big|\le2C_\varrho M_Y\Big(|s-t|^{\frac{q_1}{2}}\vee|s-t|^{q_2-\frac{q_1}{2}}\Big)+\widehat{\rho}_Y\big(|s-t|\big)+\widehat{\rho}_Y\big(\delta'_{t,s}(\omega)\big)\vee\widehat{\overline{\rho}}_Y\big(\delta'_{t,s}(\omega)\big),\end{aligned}$$

where $\delta'_{t,s}(\omega):=(1+\kappa_\wp)\Big(|s-t|^{\frac{q_1}{2}}+\phi^\omega_T\big(|s-t|\big)\Big)$. Adding them up, one can deduce from the Lipschitz continuity of random variable $\widehat{\theta}_n$ that $Z_{\widehat{\theta}_n}$ is a continuous random variable on Ω. □

Proof of (6.53)**:** If $\wp_n(\omega_1)\wedge\wp_n(\omega_2)+2^{-k}>t_1$, one has $H_{t_1}(\omega_1)=H_{t_1}(\omega_2)=0$. On the other hand, suppose that $\wp_n(\omega_1)\wedge\wp_n(\omega_2)+2^{-k}\le t_1$. When $\|\omega_1-\omega_2\|_{0,t_1}\ge2^{-k}\kappa_n^{-1}$, we automatically have $\big|H_{t_1}(\omega_1)-H_{t_1}(\omega_2)\big|\le1\le 2^k\kappa_n\|\omega_1-\omega_2\|_{0,t_1}$; When $\|\omega_1-\omega_2\|_{0,t_1}<2^{-k}\kappa_n^{-1}$, since $\wp_n(\omega_1)\wedge\wp_n(\omega_2)+\kappa_n\|\omega_1-\omega_2\|_{0,t_1}<\wp_n(\omega_1)\wedge\wp_n(\omega_2)+2^{-k}\le t_1$, applying Proposition 5.1 (2) with $t_0=t_1$ yields that $\big|\wp_n(\omega_1)-\wp_n(\omega_2)\big|\le\kappa_n\|\omega_1-\omega_2\|_{0,t_1}$. Then (2.5) implies that

$$\big|H_{t_1}(\omega_1)-H_{t_1}(\omega_2)\big|\le\big|\big(2^k(t_1-\wp_n(\omega_1))-1\big)^+-\big(2^k(t_1-\wp_n(\omega_2))-1\big)^+\big|\le2^k\big|\wp_n(\omega_1)-\wp_n(\omega_2)\big|\le2^k\kappa_n\|\omega_1-\omega_2\|_{0,t_1}.\quad\square$$

Proof of (6.67)**:** Let $\omega'\in\Omega$. If the set $\{t'\in[0,T]:\mathscr{X}(t',\omega')\le0\}$ is not empty, Proposition 5.1 (1) implies that $\lim\limits_{n\to\infty}\uparrow\wp_n(\omega')=\tau_0(\omega')$, however, $\wp_n(\omega')<\tau_0(\omega')$ for any $n\in\mathbb{N}$. Then one can deduce that $\lim\limits_{n\to\infty}\mathbf{1}_{[0,\wp_n(\omega')]}(t')=\mathbf{1}_{[0,\tau_0(\omega'))}(t')$, $\forall\,t'\in[0,T]$, and the continuity of the path $U_\cdot(\omega')$ implies that

$$\begin{aligned}\lim_{n\to\infty}\mathscr{Y}^n_{t'}(\omega') &= \lim_{n\to\infty}\big(\mathbf{1}_{\{t'\le\wp_n(\omega')\}}L(t',\omega')+\mathbf{1}_{\{t'>\wp_n(\omega')\}}U(\wp_n(\omega'),\omega')\big)\\ &= \mathbf{1}_{\{t'<\tau_0(\omega')\}}L(t',\omega')+\mathbf{1}_{\{t'\ge\tau_0(\omega')\}}U(\tau_0(\omega'),\omega')=\mathscr{Y}(\tau_0(\omega')\wedge t',\omega')=\widehat{\mathscr{Y}_{t'}}(\omega'),\quad\forall\,t'\in[0,T].\end{aligned}$$

On the other hand, if the set $\{t'\in[0,T]:\mathscr{X}(t',\omega')\le0\}$ is empty, the continuity of path $\mathscr{X}_\cdot(\omega')$ implies that $\inf\limits_{t'\in[0,T]}\mathscr{X}(t',\omega')>0$. For large enough $n\in\mathbb{N}$, the set $\big\{t'\in[0,T]:\mathscr{X}(t',\omega')\le\big(\lceil\log_2(n+2)\rceil+\lfloor\mathscr{X}_0^{-1}\rfloor-1\big)^{-1}\big\}$ is also empty, thus $T=\tau_n(\omega')=\wp_n(\omega')=\tau_0(\omega')$ by Proposition 5.1 (1). Then (A2) shows that for any $t'\in[0,T]$

$$\begin{aligned}&\lim_{n\to\infty}\mathscr{Y}^n_{t'}(\omega')=\lim_{n\to\infty}\big(\mathbf{1}_{\{t'\le\wp_n(\omega')\}}L(t',\omega')+\mathbf{1}_{\{t'>\wp_n(\omega')\}}U(\wp_n(\omega'),\omega')\big)=\mathbf{1}_{\{t'\le T\}}L(t',\omega')\\ &\quad=\mathbf{1}_{\{t'<T\}}L(t',\omega')+\mathbf{1}_{\{t'=T\}}U(T,\omega')=\mathbf{1}_{\{t'<\tau_0(\omega')\}}L(t',\omega')+\mathbf{1}_{\{t'\ge\tau_0(\omega')\}}U(\tau_0(\omega'),\omega')=\widehat{\mathscr{Y}_{t'}}(\omega').\quad\square\end{aligned}$$

Proof of (6.73)**:** Let $\omega\in\Omega$. Since $\widehat{Y}^{\ell,\ell}_t(\omega)=L_t(\omega)$ over $[0,\wp_\ell(\omega)+2^{-\ell}]\supset[0,\wp_n)$, one has

$$\widehat{\zeta}^\alpha_{i,\ell}\wedge\wp_n=\inf\big\{t\in[0,\wp_n):Z^{\ell,\ell}_t\le\widehat{Y}^{\ell,\ell}_t+1/i+1/\alpha\big\}\wedge\wp_n=\inf\big\{t\in[0,\wp_n):Z^{\ell,\ell}_t\le L_t+1/i+1/\alpha\big\}\wedge\wp_n=\zeta^\alpha_{i,\ell}\wedge\wp_n. \tag{A.16}$$

If $Z^{m_j,m_j}_t(\omega)=L_t(\omega)$ for some $t\in\big[0,\wp_n(\omega)\big)$, applying (6.70) with $k=m_j$ and $k=\ell$ respectively shows that $\mathscr{Z}_t(\omega)\le Z^{m_j,m_j}_t(\omega)+\overline{\varepsilon}_{m_j}\le L_t(\omega)+\overline{\varepsilon}_\ell<L_t(\omega)+\frac{1}{2i}+\frac{1}{\alpha}$ and thus $Z^{\ell,\ell}_t(\omega)\le\mathscr{Z}_t(\omega)+\overline{\varepsilon}_\ell<L_t(\omega)+\frac{1}{i}+\frac{1}{\alpha}$. So $\inf\{t\in[0,\wp_n(\omega)):\ Z^{\ell,\ell}_t(\omega)\le L_t(\omega)+1/i+1/\alpha\}\le\inf\{t\in[0,\wp_n(\omega)):\ Z^{m_j,m_j}_t(\omega)=L_t(\omega)\}$. As $\widehat{Y}^{m_j,m_j}_t(\omega)=L_t(\omega)$ over $[0,\wp_n(\omega))$, one can deduce that

$$\begin{aligned}\zeta^\alpha_{i,\ell}(\omega)\wedge\wp_n(\omega)&=\inf\big\{t\in[0,\wp_n(\omega)):\ Z^{\ell,\ell}_t(\omega)\le L_t(\omega)+1/i+1/\alpha\big\}\wedge\wp_n(\omega)\le\inf\{t\in[0,\wp_n(\omega)):\ Z^{m_j,m_j}_t(\omega)=L_t(\omega)\}\wedge\wp_n(\omega)\\&=\inf\{t\in[0,\wp_n(\omega)):\ Z^{m_j,m_j}_t(\omega)=\widehat{Y}^{m_j,m_j}_t(\omega)\}\wedge\wp_n(\omega)=\nu_{m_j}(\omega)\wedge\wp_n(\omega).\end{aligned}$$

On the other hand, if the set $\big\{t\in[0,\wp_n(\omega)):\ Z^{m_j,m_j}_t(\omega)=L_t(\omega)\big\}$ is empty, we can deduce that $\nu_{m_j}(\omega)\ge\wp_n(\omega)$. Then $\nu_{m_j}(\omega)\wedge\wp_n(\omega)=\wp_n(\omega)\ge\zeta^\alpha_{i,\ell}(\omega)\wedge\wp_n(\omega)$ holds automatically. □

Proof of (6.79)**:** Set $\widehat{\zeta}'_{i,\ell}:=\lim\limits_{\alpha\to\infty}\uparrow\widehat{\zeta}^\alpha_{i,\ell}\le\widehat{\zeta}_{i,\ell}$. As the continuity of $Z^{\ell,\ell}-\widehat{Y}^{\ell,\ell}$ shows that $Z^{\ell,\ell}_{\widehat{\zeta}^\alpha_{i,\ell}}-\widehat{Y}^{\ell,\ell}_{\widehat{\zeta}^\alpha_{i,\ell}}\le\frac{1}{i}+\frac{1}{\alpha}$, $\forall\,\alpha>\ell$, letting $\alpha\to\infty$, we see from the continuity of $Z^{\ell,\ell}-\widehat{Y}^{\ell,\ell}$ again that $Z^{\ell,\ell}_{\widehat{\zeta}'_{i,\ell}}-\widehat{Y}^{\ell,\ell}_{\widehat{\zeta}'_{i,\ell}}\le1/i$, which implies that $\widehat{\zeta}_{i,\ell}=\widehat{\zeta}'_{i,\ell}=\lim\limits_{\alpha\to\infty}\uparrow\widehat{\zeta}^\alpha_{i,\ell}$. □

Proof of (6.82)**:** Let $\omega\in\Omega$. If the set $\mathcal{I}(\omega):=\big\{t\in[0,T]:\ \mathscr{Z}_t(\omega)\le L_t(\omega)+1/i\big\}$ is empty, the continuity of path $\mathscr{Z}_\cdot(\omega)-L_\cdot(\omega)$ implies that $\eta(\omega):=\inf\limits_{t\in[0,T]}\big(\mathscr{Z}_t(\omega)-L_t(\omega)\big)>1/i$. For any integer $h>(\eta(\omega)-1/i)^{-1}$, since $\inf\limits_{t\in[0,T]}(\mathscr{Z}_t(\omega)-L_t(\omega))=\eta(\omega)>1/i+1/h$, the set $\mathcal{I}_h(\omega):=\big\{t\in[0,T]:\ \mathscr{Z}_t(\omega)\le L_t(\omega)+1/i+1/h\big\}$ is also empty and thus $\gamma^h_i(\omega)=T$. It follows that $\lim\limits_{h\to\infty}\uparrow\gamma^h_i(\omega)=T=\gamma_i(\omega)$.

On the other hand, if $\mathcal{I}(\omega)$ is not empty, we set $\gamma'_i(\omega):=\lim\limits_{h\to\infty}\uparrow\gamma^h_i(\omega)\le\gamma_i(\omega)=\inf\mathcal{I}(\omega)$. For any $h\in\mathbb{N}$, $\mathcal{I}_h(\omega)$ contains $\mathcal{I}(\omega)$ and is thus not empty. The continuity of path $\mathscr{Z}_\cdot(\omega)-L_\cdot(\omega)$ then implies that $\mathscr{Z}\big(\gamma^h_i(\omega),\omega\big)-L\big(\gamma^h_i(\omega),\omega\big)\le\frac{1}{i}+\frac{1}{h}$. Letting $h\to\infty$, we see from the continuity of path $\mathscr{Z}_\cdot(\omega)-L_\cdot(\omega)$ again that $\mathscr{Z}\big(\gamma'_i(\omega),\omega\big)-L\big(\gamma'_i(\omega),\omega\big)\le1/i$, which shows that $\gamma_i(\omega)=\inf\mathcal{I}(\omega)\le\gamma'_i(\omega)$. Thus $\gamma_i(\omega)=\gamma'_i(\omega)=\lim\limits_{h\to\infty}\uparrow\gamma^h_i(\omega)$. □

Proof of (6.89)**:** Set $s_n=s_n(\omega):=(\gamma_*\wedge\wp_n)(\omega)$ and let $\varepsilon>0$. By the continuity of path $\mathscr{Z}_\cdot(\omega)$, there exists a $\delta_n=\delta_n(\omega)>0$ such that $\big|\mathscr{Z}_t(\omega)-\mathscr{Z}\big(s_n,\omega\big)\big|\le\varepsilon$, $\forall\,t\in\big[(s_n-\delta_n)^+,s_n\big]$. We see from (6.88) that for large enough $i\in\mathbb{N}$, both $(\gamma_i\wedge\wp_n)(\omega)$ and $(\gamma_{2i}\wedge\wp_n)(\omega)$ are in $\big[(s_n-\delta_n)^+,s_n\big]$, so $\mathcal{J}_{n,i}(\omega)\subset\big[(s_n-\delta_n)^+,s_n\big]$. It follows that $\mathscr{Z}\big(s_n,\omega\big)-\varepsilon\le\inf\limits_{t\in\mathcal{J}_{n,i}(\omega)}\mathscr{Z}(t,\omega)\le\sup\limits_{t\in\mathcal{J}_{n,i}(\omega)}\mathscr{Z}(t,\omega)\le\mathscr{Z}\big(s_n,\omega\big)+\varepsilon$. As $i\to\infty$, we obtain $\mathscr{Z}\big(s_n,\omega\big)-\varepsilon\le\varliminf\limits_{i\to\infty}\inf\limits_{t\in\mathcal{J}_{n,i}(\omega)}\mathscr{Z}(t,\omega)\le\varlimsup\limits_{i\to\infty}\sup\limits_{t\in\mathcal{J}_{n,i}(\omega)}\mathscr{Z}(t,\omega)\le\mathscr{Z}\big(s_n,\omega\big)+\varepsilon$. Letting $\varepsilon\to0$ then yields to (6.89). □

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